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Wave-Action Conservation for Pseudo-Hermitian Fields

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The linearized Vlasov-Maxwell system and ideal magnetohydrodynamics are examples of systems with a *pseudo-Hermitian* structure with respect to an *indefinite* inner product on the space of complex representations of real fields [J. Larsson, Phys. Rev. Lett. **66**, 1466 (1991)]. This structure leads to variational principles, whose variation with respect to local wave phase yields new wave-action conservation laws, which generalize previous eikonal versions.

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For some years, the law of wave-action conservation for classical fields has been based on the *eikonal* approximation, which requires that wave amplitudes are slowly varying spatially on the scale of the wave phase [1–3]. Yet there are important situations in physics where this approximation is clearly invalid, and where intuition leads one still to expect action conservation. The need for such a conservation law is especially important for wave propagation in a medium with weak time dependence, where wave-energy conservation is lost. Some common situations where eikonality fails are the following: (1) Waves propagate across *discontinuities*, undergoing refraction and partial reflection; (2) in plasma kinetic models, wave action is transferred between collective and kinetic modes, often over a *singular layer* [4–6]; and (3) waves are *scattered* as they propagate through a turbulent medium [7].

Recently Kull, Berk, and Morrison [8] presented a noneikonal formulation of wave-action conservation for a field whose evolution is governed by a *Hermitian* operator. This requirement of Hermiticity restricts its applicability by omitting effects such as wave-particle resonance in Vlasov theory. However, there is an important class of systems whose wave operators are *pseudo-Hermitian* [9], in the sense that Hermiticity is defined with respect to an *indefinite*-metric inner product [10].

Several important dynamical systems have a *Hamiltonian* structure in terms of a generalized (noncanonical) Poisson bracket [11]. The best-known examples in plasma physics are ideal magnetohydrodynamics (MHD) [12] and the Vlasov-Maxwell model of collisionless plasma dynamics [13]. In the process of linearization, the first-order fields represent perturbations from an arbitrary (possibly time-dependent) *reference* state. These fields can be expressed in terms of generating functions; in linearized ideal MHD theory, for example, the linear fields $(\rho_1, \mathbf{u}_1, p_1, \mathbf{B}_1)$ are expressed in terms of the fluid displacement $\boldsymbol{\xi}(\mathbf{x}, t)$ [e.g., $\rho_1 = -\,\boldsymbol{\nabla}\cdot(\rho_0\boldsymbol{\xi})$].

For the Vlasov-Maxwell system, Larsson [9] developed the linearization in terms of $S(z;t)$, a scalar generating function for canonical transformations on six-dimensional phase space $z \equiv (\mathbf{r}, \mathbf{p} \equiv m\mathbf{v})$. The linear perturbation of the Vlasov distribution, f_1, is generated jointly by S and $\mathbf{A}_1(\mathbf{x}, t)$ (the linearized vector potential) acting on the reference distribution $f_0(z;t)$: $f_1 = \{S, f_0\} + (e/c)\mathbf{A}_1(\mathbf{r}, t)\cdot\partial f_0/\partial\mathbf{p}$. This represents dynamically accessible perturbations [14–16]. Larsson showed that, by *complexifying* the linear fields, and by defining an indefinite-metric inner product (with zero norm for real fields), the time-evolution operator was indeed Hermitian with respect to this unconventional inner product. Upon our recognition that this pseudo-Hermiticity was a general property of linearized Hamiltonian systems [17], one of us developed the appropriate formulation for linearized MHD [18].

An immediate consequence of the pseudo-Hermitian formulation is its associated quadratic *variational principle*. In this paper we show how to use this principle to construct the wave-action continuity equation, by considering an arbitrary *nonuniform* phase shift of the complexified wave field. The three examples we treat are the following: (1) ideal MHD, of which fluid dynamics is a special case; (2) cold multifluid electrodynamics, which also has a noncanonical Hamiltonian structure; and (3) the Vlasov-Maxwell system, of which the Vlasov-Coulomb system is a special case. In each case, the reference state (about which linearization is taken) is allowed to be time dependent.

The expressions obtained for the wave-action density and flux are local quadratic functions of the *complexified* wave fields [see, e.g., Eqs. (4), (8), and (15)]. Complex representations have been common for real linear fields, but have usually been considered merely an algebraic convenience. Here, however, complexification is *essential*, as these expressions vanish identically for real fields. [One accepted algorithm for complexification is Gabor's analytic signal [19]: For any real function of time $f(t)$, define the analytic $f_A(t) \equiv \int_0^\infty (d\omega/\pi)\, e^{-i\omega t} \bar{f}(\omega)$, with $\bar{f}(\omega) \equiv \int_{-\infty}^{+\infty} dt\, e^{+i\omega t} f(t)$. The inverse is simply $f(t) = \mathrm{Re}\, f_A(t)$.]

Our derivation of the wave-action continuity equation

$$\partial_t J(\mathbf{x};t) + \nabla \cdot \mathbf{J}(\mathbf{x};t) = 0, \tag{1}$$

for the action density J and the action flux density $\mathbf{J}$, is based on a variational functional $\mathcal{A}$ for the complexified wave field. Because $\mathcal{A}$ is stationary for *all* infinitesimal variations of the field about a solution, it is stationary for the set of restricted variations representing infinitesimal *nonuniform* phase shifts. This is the generalization of the space-time variation of the eikonal phase in the standard derivations for eikonal wave fields. Formally, (1) states that the functional derivative of $\mathcal{A}$ with respect to the wave phase vanishes.

For each of the three examples presented in this paper, we have explicitly evaluated $\partial J/\partial t$ by using (4), (8), or (15), the evolution equations for an *arbitrary* reference state, and the evolution equations for the wave field. In each case, we have rigorously verified that Eq. (1) holds with $\mathbf{J}$ given by (5), (9), or (16), respectively, with *no* approximation.

We illustrate the method by considering first the case of ideal single-fluid MHD, where the wave field is represented by the fluid displacement $\boldsymbol{\xi}(\mathbf{x},t)$. The variational functional, yielding the standard evolution equation for $\boldsymbol{\xi}$ [but allowing for *time-dependent* reference state $\mathbf{B}_0(\mathbf{x},t)$, $\rho_0(\mathbf{x},t)$, $\mathbf{u}_0(\mathbf{x},t)$, $p_0(\mathbf{x},t)$], is [18, 20]

$$\mathcal{A}(\boldsymbol{\xi}) \equiv \frac{1}{2}\int d^4x \,\{\rho_0|\dot{\boldsymbol{\xi}}|^2 + \boldsymbol{\xi}^* \cdot [\nabla \cdot \mathsf{K}(\boldsymbol{\xi})]\}\ , \tag{2}$$

in terms of the Hermitian operator $\nabla \cdot \mathsf{K}$, where

$$\mathsf{K}(\boldsymbol{\xi}) \equiv -(p_1 + \mathbf{B}_0 \cdot \mathbf{Q})\,\mathsf{I} + \mathbf{B}_0\mathbf{Q} + \mathbf{Q}\mathbf{B}_0 + \boldsymbol{\xi}\rho_0\dot{\mathbf{u}}_0\ ,$$

$\mathbf{Q}(\boldsymbol{\xi}) \equiv \nabla \times (\boldsymbol{\xi} \times \mathbf{B}_0)/4\pi \equiv \mathbf{B}_1/4\pi$, $p_1(\boldsymbol{\xi}) \equiv -\boldsymbol{\xi}\cdot\nabla p_0 - \gamma p_0 \nabla\cdot\boldsymbol{\xi}$, $\dot{\boldsymbol{\xi}} \equiv \partial\boldsymbol{\xi}/\partial t + \mathbf{u}_0\cdot\nabla\boldsymbol{\xi}$. In (2), we take the variation of $\mathcal{A}$ for $\boldsymbol{\xi}(\mathbf{x},t) \to \boldsymbol{\xi}(\mathbf{x},t)\exp i\alpha(\mathbf{x},t)$, with α infinitesimal and arbitrary; thus $\delta\boldsymbol{\xi} = i\alpha\boldsymbol{\xi}$, $\delta\boldsymbol{\xi}^* = -i\alpha\boldsymbol{\xi}^*$. After some integration by parts (letting α have compact support), and using the Hermiticity of K, we find

$$\delta\mathcal{A} = \int d^4x\, \alpha(\mathbf{x},t)[\partial J/\partial t + \nabla\cdot\mathbf{J}], \tag{3}$$

with the action density

$$J(\mathbf{x};t) \equiv \mathrm{Im}\,[\rho_0(\mathbf{x},t)\dot{\boldsymbol{\xi}}^*(\mathbf{x},t)\cdot\boldsymbol{\xi}(\mathbf{x},t)], \tag{4}$$

and the action flux density

$$\mathbf{J}(\mathbf{x};t) \equiv \mathbf{J}^{(0)}(\mathbf{x};t) + \mathbf{u}_0(\mathbf{x},t)J(\mathbf{x};t), \tag{5}$$

where

$$\mathbf{J}^{(0)}(\mathbf{x};t) \equiv \mathrm{Im}\,[\,(\boldsymbol{\xi}^* \times \mathbf{B}_0)\times\mathbf{Q} - \boldsymbol{\xi}^* p_1 - (\boldsymbol{\xi}^*\times\boldsymbol{\xi}\cdot\mathbf{j}_0)\mathbf{B}_0/2c] \tag{6}$$

is the action flux density in the local rest frame, and $\mathbf{u}_0 J$ is the convective contribution. Because $\delta A = 0$ for all α, we obtain the desired action continuity equation (1).

The expressions (4) and (5) have the following properties: (a) For real $\boldsymbol{\xi}$, they vanish *identically*; (b) because $\dot{\boldsymbol{\xi}}$ is the *convective* time derivative, J is invariant under Galilean frame changes; and (c) in the *eikonal* limit, $\dot{\boldsymbol{\xi}} \to -i(\omega - \mathbf{k}\cdot\mathbf{u}_0)\boldsymbol{\xi}$, $J(\mathbf{x},t) \to (\omega - \mathbf{k}\cdot\mathbf{u}_0)\rho_0|\boldsymbol{\xi}|^2$, and $\mathbf{J} = J\times$(local group velocity), the standard results in MHD [2].

When the magnetic terms are deleted from the MHD formulas, one obtains results appropriate for fluid dynamics. They can also be generalized in several directions: to rotating frames [21], to dielectric fluids, and to special-relativistic [22, 23] and general-relativistic [24] fluids.

We next consider ideal cold multispecies electrodynamics [25]. The first-order fields are $(\boldsymbol{\xi}_s, \phi_1, \mathbf{A}_1)$, where $\boldsymbol{\xi}_s$ denotes the displacement for species s, and $(\phi_1, \mathbf{A}_1)$ are the first-order electromagnetic potentials. The reference state is represented by the (time-dependent) fields: n_0^s, $\mathbf{u}_0^s$, $\mathbf{E}_0$, $\mathbf{B}_0$. The variational functional can be derived from the cold-fluid limit of the Low Lagrangian [26]:

$$\begin{aligned}\mathcal{A}(\phi_1,\mathbf{A}_1,\boldsymbol{\xi}_s) \equiv\ & \int d^4x \left[\frac{|\mathbf{E}_1|^2 - |\mathbf{B}_1|^2}{8\pi} + \sum_s \frac{n_0^s m_s}{2}|\dot{\boldsymbol{\xi}}_s|^2\right] \\ & + \mathrm{Re}\int d^4x \sum_s n_0^s e_s\{\boldsymbol{\xi}_s^* \times \dot{\boldsymbol{\xi}}_s\cdot\mathbf{B}_0/2c + \tfrac{1}{2}\boldsymbol{\xi}_s^*\cdot[\nabla\mathbf{E}_0 - (\nabla\mathbf{B}_0)\times\mathbf{u}_0^s/c]\cdot\boldsymbol{\xi}_s \\ & + \boldsymbol{\xi}_s^*\cdot(\mathbf{E}_1 + \mathbf{u}_0^s\times\mathbf{B}_1/c)\},\end{aligned} \tag{7}$$

with $\mathbf{E}_1 \equiv -\nabla\phi_1 - c^{-1}\partial\mathbf{A}_1/\partial t$, and $\mathbf{B}_1 \equiv \nabla\times\mathbf{A}_1$. By the same method as for MHD, using the *same* α for ϕ_1, $\mathbf{A}_1$, $\boldsymbol{\xi}_s$ ($\delta\phi_1 = i\alpha\phi_1$, $\delta\mathbf{A}_1 = i\alpha\mathbf{A}_1$, $\delta\boldsymbol{\xi}_s = i\alpha\boldsymbol{\xi}_s$), we derive the action density

$$J(\mathbf{x};t) \equiv \mathrm{Im}\left\{\mathbf{A}_1^*\cdot\mathbf{E}_1/4\pi c + \sum_s \frac{n_0^s e_s}{2c}\mathbf{B}_0\cdot\boldsymbol{\xi}_s^*\times\boldsymbol{\xi}_s - \sum_s n_0^s\boldsymbol{\xi}_s^*\cdot\left(m_s\dot{\boldsymbol{\xi}}_s + \frac{e_s}{c}\mathbf{A}_1\right)\right\} \tag{8}$$

and the action flux density

$$\mathbf{J}(\mathbf{x};t) \equiv \mathrm{Im}\left((\phi_1^*\mathbf{E}_1 + \mathbf{A}_1^*\times\mathbf{B}_1)/4\pi + \sum_s n_0^s\left\{\mathbf{u}_0^s\left[-\boldsymbol{\xi}_s^*\cdot\left(m_s\dot{\boldsymbol{\xi}}_s + \frac{e_s}{2c}\mathbf{A}_1\right) + \frac{e_s}{2c}\mathbf{B}_0\cdot\boldsymbol{\xi}_s^*\times\boldsymbol{\xi}_s\right] - e_s\boldsymbol{\xi}_s^*\left(\phi_1 - \frac{\mathbf{u}_0^s}{c}\cdot\mathbf{A}_1\right)\right\}\right). \tag{9}$$

In the *eikonal* limit $\mathbf{A}_1 \equiv \tilde{\mathbf{A}}(\mathbf{x},t)\exp i\Theta(\mathbf{x},t)$, the species displacements $\boldsymbol{\xi}_s(\mathbf{x},t)$ can be algebraically expressed in terms of the local linear electromagnetic field. Then the action density reduces (in the radiation gauge $\phi_1 \equiv 0$) to the standard expression [3]

$$J = \tilde{\mathbf{A}}^*\cdot(\partial\mathsf{D}/\partial\omega)\cdot\tilde{\mathbf{A}}, \tag{10}$$

and the action flux density to

$$\mathbf{J} = -\tilde{\mathbf{A}}^*\cdot(\partial\mathsf{D}/\partial\mathbf{k})\cdot\tilde{\mathbf{A}}, \tag{11}$$

where $\mathsf{D}(\mathbf{k},\omega;\mathbf{x},t)$ is the local *cold-plasma* dispersion tensor (allowing for nonzero flow $\mathbf{u}_0^s$). On introducing the local polarization by $\tilde{\mathbf{A}} = \hat{\mathbf{e}}(\mathbf{x},t)A(\mathbf{x},t)$, these reduce further to the standard forms

$$J = (\partial D/\partial\omega)|A|^2, \tag{12}$$

$$\mathbf{J} = J\times(\text{local group velocity}), \tag{13}$$

where $D \equiv \hat{\mathbf{e}}^*\cdot\mathsf{D}\cdot\hat{\mathbf{e}}$. Also [3], in the eikonal limit, the wave-energy density is ωJ, and the wave-momentum density is $\mathbf{k}J$.

Finally we turn to the Vlasov-Maxwell model of collisionless plasma dynamics. Larsson [27] has recently derived the variational functional for the Vlasov-Coulomb model from the Low Lagrangian [26]. The generalization to the Vlasov-Maxwell model is straightforward; the first-order fields are the generating function $S(z;t)$ and the electromagnetic potentials $\phi_1(x)$ and $\mathbf{A}_1(x)$. The variational functional is

$$\mathcal{A}(\mathbf{A}_1,\phi_1,S) \equiv \int d^4x\,\left(|\mathbf{E}_1|^2 - |\mathbf{B}_1|^2\right)/8\pi - \frac{1}{2}\int dt\int d^6z\, f_0(z;t)\left[e^2|\mathbf{A}_1|^2/mc^2 + \mathrm{Re}\{S^*,\dot{S} - 2e(\phi_1 - \mathbf{v}\cdot\mathbf{A}_1/c)\}\right], \tag{14}$$

where

$$\dot{S} \equiv \frac{\partial S}{\partial t} + \{S,h_0\} - \frac{e}{c}\frac{\partial\mathbf{A}_0}{\partial t}\cdot\frac{\partial S}{\partial\mathbf{p}},$$

and

$$h_0(z,t) \equiv p^2/2m + e\phi_0(\mathbf{r},t),$$

summation over species is implicit, and the noncanonical Poisson bracket [28] is

$$\{f,g\} \equiv \frac{\partial f}{\partial\mathbf{r}}\cdot\frac{\partial g}{\partial\mathbf{p}} - \frac{\partial f}{\partial\mathbf{p}}\cdot\frac{\partial g}{\partial\mathbf{r}} + \frac{e}{c}\mathbf{B}_0(\mathbf{r},t)\cdot\frac{\partial f}{\partial\mathbf{p}}\times\frac{\partial g}{\partial\mathbf{p}},$$

in terms of the possibly time-dependent *reference* field $\mathbf{B}_0$.

We proceed as before and obtain for the action density

$$J(\mathbf{x};t) \equiv \mathrm{Im}\left[\mathbf{A}_1^*\cdot\mathbf{E}_1/4\pi c + \frac{1}{2}\int d^6z\,\delta^3(\mathbf{x}-\mathbf{r})\,\{S^*,f_0\}S\right], \tag{15}$$

and the action flux density is

$$\mathbf{J}(\mathbf{x};t) \equiv \mathrm{Im}\left[(\phi_1^*\mathbf{E}_1 + \mathbf{A}_1^*\times\mathbf{B}_1)/4\pi + \frac{1}{2}\int d^6z\,\delta^3(\mathbf{x}-\mathbf{r})\left(\{S^*,f_0\}\mathbf{v} - e(\phi_1^* - \mathbf{v}\cdot\mathbf{A}_1^*/c)\frac{\partial f_0}{\partial\mathbf{p}}\right)S\right], \tag{16}$$

where we have used $\dot{S} = e(\phi_1 - \mathbf{v}\cdot\mathbf{A}_1/c)$, which follows from the full variation. We obtain the expressions for J and $\mathbf{J}$ for the linearized Vlasov-Coulomb model by setting $\mathbf{A}_1 = 0$.

Just as for the cold-fluid model, one can take the eikonal limit, and then express only the *nonresonant* part of S in terms of the linear electromagnetic field. The local dispersion tensor is then Hermitian (in the conventional sense), while the *resonant* part of S appears explicitly. This formulation is useful in treating linear conversion between collective and ballistic modes [4–6].

In conclusion, we have derived wave-action conservation laws for several linear-wave systems, in terms of complex representations of the wave fields, which evolve by pseudo-Hermitian operators. The general procedure is as follows: From the variational functional $\mathcal{A}(\psi^*, \psi)$ in terms of the complexified field ψ, we perform the *restricted* variation $\delta\psi = i\alpha\psi$ and $\delta\psi^* = -i\alpha\psi^*$, and obtain the functional $\Delta\mathcal{A}(\psi^*, \psi; \alpha) \equiv \mathcal{A}(-i\alpha\psi^*, \psi) + \mathcal{A}(\psi^*, +i\alpha\psi)$. By construction, any terms in $\mathcal{A}$ which do not involve space-time derivatives on ψ will not contribute to the wave-action conservation law. Then the action density $J(\mathbf{x}, t)$ is the functional derivative of $\Delta\mathcal{A}$ with respect to $\partial\alpha(\mathbf{x}, t)/\partial t$, while the flux $\mathbf{J}$ is the functional derivative with respect to $\boldsymbol{\nabla}\alpha$.

Because these wave-action conservation laws do not require eikonality, one can expect many applications, such as those listed in the opening paragraph.

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