Universal Velocity Correlations in Disordered and Chaotic Systems

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The response of weakly disordered metallic grains to Aharonov-Bohm flux suggests a rescaling in which statistical correlators become universal. We derive an exact expression for a correlator of level "velocities," and provide numerical evidence which suggests that the universality extends to a wider class of systems and generalizes to arbitrary perturbations providing a new characterization of quantum chaos. These results are applied to Fermi velocities of complex lattices.

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The Wigner-Dyson distribution of level spacings [1] provides an accurate description of a variety of complex systems ranging from those with many degrees of freedom and strong interactions (e.g., atomic nuclei) to the quantum mechanical motion of particles in irregular potentials (e.g., quantum dots, disordered metallic grains). The distribution serves as a universal classification of quantum chaos [2]. With such correlations, the dependence on the system enters only through the mean level spacing, Δ. Here we provide exact analytical results which suggest that the response of the energy levels E_i to an external perturbation, controlled by some parameter X [3–5], relies on just one additional parameter,

$$C(0) = \frac{1}{\Delta^2}\Big\langle\Big(\frac{\partial E_i(X)}{\partial X}\Big)^2\Big\rangle, \tag{1}$$

where $\langle\cdots\rangle$ denotes a statistical average which, for example, can be performed over a typical range of levels, and/or X. We propose that after rescaling,

$$x = \sqrt{C(0)}X, \qquad \epsilon_i(x) = E_i(X)/\Delta, \tag{2}$$

the statistical properties of the random functions $\epsilon_i(x)$ *are universal* independent of the nature of X. The physical interpretation of $C(0)$ as a "generalized conductance" can be found through a universal fluctuation-dissipation theorem [3, 6] which can be derived for this class of system. The nonuniversality of $C(0)$ prevents a more precise general definition.

We will seek to establish this rescaling by focusing on a particular class of chaotic system in which the averaging can be performed from a statistical ensemble. For disordered systems it is possible not only to motivate the rescaling of Eq. (2) but to derive an exact expression for the autocorrelator of density of states (DOS) fluctuations, from which we can determine, for example, the autocorrelator of level "velocities" $\partial_x\epsilon_i(x) \equiv \partial\epsilon_i(x)/\partial x$,

$$\widetilde{c}(\omega, x) = \frac{\sum_{ij}\langle\delta(\epsilon_i(\bar{x}) - \epsilon_j(\bar{x}+x) - \omega)\partial_{\bar{x}}\epsilon_i(\bar{x})\ \partial_{\bar{x}}\epsilon_j(\bar{x}+x)\rangle}{\sum_{ij}\langle\delta(\epsilon_i(\bar{x}) - \epsilon_j(\bar{x}+x) - \omega)\rangle}. \tag{3}$$

The analytical expression for $\widetilde{c}(\omega, x)$ will be checked by numerical simulation of an Anderson model of disorder, and the universality confirmed by a chaotic billiard.

If we chose the perturbation X to be an Aharonov-Bohm flux through a ring, $\widetilde{c}(\omega, x)$ acquires the following interpretation. The flux ϕ acts as a quasiperiodic boundary condition, so that $E_i(\phi)$ describes the dispersion of a periodic one-dimensional lattice with the open ring defining the unit cell. Higher dimensions of the lattice d_x require the generalization to a torus with components of "flux," ϕ_μ $(1 \le \mu \le d_x)$ acting along all spatial directions. The quasimomentum of the Bloch function $\boldsymbol{\pi} = 2\pi\boldsymbol{\phi}/L$, where L denotes the sample (unit cell) size and flux is measured in units of the flux quantum $\phi_0 = hc/e$. With $\mathbf{X} \equiv \boldsymbol{\pi}$ [7], $\widetilde{c}(\omega, x)$ measures the average autocorrelation of velocities at the Fermi surface. Equation (2) implies that, after proper rescaling (and removal of discrete symmetries), the autocorrelation of Fermi velocities of complex crystalline lattices (e.g., antidot arrays) becomes universal.

Consider the Hamiltonian describing the motion of a spinless particle confined to a d-torus and subject to an Aharonov-Bohm flux $\boldsymbol{\phi}$ and δ-correlated disorder potential,

$$H(\phi_\mu) = \frac{1}{2m}\sum_\mu (p_\mu - h\phi_\mu/L)^2 + W(\mathbf{r}). \tag{4}$$

We assume that $2/mv^2 \ll \tau \ll L/v$, where v denotes the velocity of the particle on the torus, and the mean free time between collisions τ is defined through the ensemble average, $\langle W(\mathbf{r})W(\mathbf{r}')\rangle = L^d\Delta\delta(\mathbf{r}-\mathbf{r}')/2\pi\tau$ [with $\langle W(\mathbf{r})\rangle = 0$]. This implies that the particle motion is diffusive, with the classical diffusion constant $D = v^2\tau/d$.

The application of diagrammatic perturbation theory to determine the dimensionless autocorrelator of DOS fluctuations,

$$k(\Omega, \boldsymbol{\phi}) = \sum_{ij}\Delta^2\langle\delta(E - \Omega - E_i(\bar{\boldsymbol{\phi}}))\delta(E - E_j(\bar{\boldsymbol{\phi}} + \boldsymbol{\phi}))\rangle - 1\,, \tag{5}$$

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leads to pathological divergences when $\phi^2 < 1/g$ [8], where $g = D/L^2\Delta$ denotes the dimensionless zero temperature conductance. Fortunately, for small ϕ we can apply the method of supersymmetry [9] to determine $k(\Omega, \phi)$ nonperturbatively. A straightforward but lengthy generalization of existing theory [10], which we outline presently, gives

$$k^u(\Omega, \phi) = \frac{1}{2}\mathrm{Re}\int_{-1}^{1} d\lambda \int_{1}^{\infty} d\lambda_1 \exp[-2\pi^3 g\phi^2(\lambda_1^2 - \lambda^2) - i\pi\Omega(\lambda - \lambda_1 + i\delta)/\Delta], \tag{6}$$

where the superscript u reflects unitary symmetry of the Hamiltonian. Equation (6), which is valid for $|\phi_\mu| < 1/2$, shows $k(\Omega, \phi)$ to depend only on g and Δ.

By using Eq. (6) we find [10]

$$C^{\mu\mu}(0) \equiv \left\langle \left(\frac{\partial_{\phi_\mu} E_i(\phi)}{\Delta} \right)^2 \right\rangle = 4\pi g_\mu, \tag{7}$$

where g_μ is the dimensionless conductance resolved along direction μ. Equation (7), which is analogous to the Thouless formula for conductance [11, 12], can be used to apply the rescaling of Eq. (2) (generalized to an arbitrary number of components),

$$x_\mu^2 = C^{\mu\mu}(0)\phi_\mu^2, \qquad \omega = \Omega/\Delta. \tag{8}$$

The result is a universal expression,

$$k^u(\omega, \mathbf{x}) = \int_0^\infty \frac{d\lambda}{2\pi^2\mathbf{x}^2\lambda}\{\exp[-\mathbf{x}^2|\pi\lambda - \lambda^2/2|] - \exp[-\mathbf{x}^2(\pi\lambda + \lambda^2/2)]\}\cos(\omega\lambda). \tag{9}$$

According to the universality, Eq. (9) should be valid for all unitary systems (broken T invariance). This can be verified for disordered metallic grains perturbed by a magnetic field $X = BL^2/\phi_0$ when $C(0) = \pi g/3$ [10]. The range of $\mathbf{x}$ over which Eq. (9) is valid is dependent on nonuniversal properties of the system. For flux, breakdown of universality occurs at $\phi \sim 1/2$ (or equivalently $x \sim \sqrt{g}$) when higher mode corrections become significant and ultimately recover the periodicity. More generally, we believe that the cutoff is set by $x \sim (t_0\Delta)^{-1/2}$, where t_0 is of the order of the time of the shortest periodic trajectories. (In disordered metals $t_0 \sim L^2/D$, while for ballistic chaotic systems, $t_0 \sim L/v$.)

When the dimensionality of the parameter space d_x [7] is greater than unity, as with multicomponent flux, we can define the generalization of Eq. (3),

$$\widetilde{c}^{\mu\eta}(\omega, \mathbf{x}) = [1 + k(\omega, \mathbf{x})]^{-1}\partial_{x_\mu}\partial_{x_\eta} \times \int_{-\infty}^{\omega} d\epsilon_1 \int_{-\infty}^{0} d\epsilon_2\, k(\epsilon_1 - \epsilon_2, \mathbf{x}). \tag{10}$$

In Fig. 1 we show $\widetilde{c}(0, \mathbf{x}) \equiv d_x^{-1}\mathrm{Tr}\ \widetilde{c}^{\mu\eta}(0, \mathbf{x})$ for $d_x = 1$, 2, and 3 with asymptotics,

$$\widetilde{c}_u(0, \mathbf{x}) = \begin{cases} 1 - 1/d_x - \mathcal{O}(\mathbf{x}^2), & \mathbf{x} \to 0, \\ (d_x - 2)/d_x\pi^2\mathbf{x}^2, & x_\mu \gg 1. \end{cases} \tag{11}$$

$\widetilde{c}(0, \mathbf{x})$ measures the averaged autocorrelation of velocities on a $(d_x - 1)$-dimensional "Fermi" surface. At $\mathbf{x} = 0$, it shows singular behavior being equal to unity at the origin but tending to $1 - 1/d_x$ as $\mathbf{x} \to 0$. For $d_x = 1$ contributions to $\widetilde{c}(0, x)$ as $x \to 0$ arise from consecutive crossings of the Fermi level close to a turning point and vanish like $-x^2$. The contribution from the first return dominates and $\widetilde{c}(0, x)$ remains negative over the whole region of x. In higher dimensions, the connected part of the surface can give nonzero contributions along $d_x - 1$ dimensions even in the limit $\mathbf{x} \to 0$, and $\widetilde{c}(0, \mathbf{x})$ remains finite approaching unity in the limit of high d_x. For $d_x = 2$ this "diagonal" contribution dominates over most of the range with $\widetilde{c}(0, \mathbf{x})$ becoming negative only at large values of $\mathbf{x}$, while for $d_x > 2$, it remains positive at all $\mathbf{x}$.

The behavior of $\widetilde{c}(\omega, x)$ at $\omega \neq 0$ in $d_x = 1$ is shown inset in Fig. 1. $\widetilde{c}(\omega, x)$ develops a maximum in x coin-

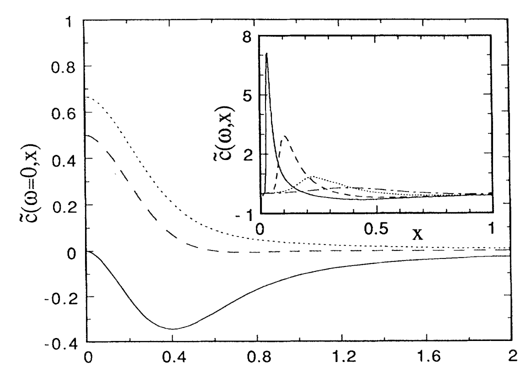

FIG. 1. Autocorrelation of velocities $\widetilde{c}(0, \mathbf{x})$ shown for $d =$ 1, 2, and 3 as a function of $|\mathbf{x}|$ for $d = 1$ (full line), $d = 2$ (dashed line), and $d = 3$ (dotted line). The variation of $\widetilde{c}(\omega, x)$ for $d = 1$ is shown inset for values of $\omega = 0.1$ (full line), $\omega = 0.25$ (dashed line), $\omega = 0.5$ (dotted line), and $\omega = 1$ (dash-dotted line).

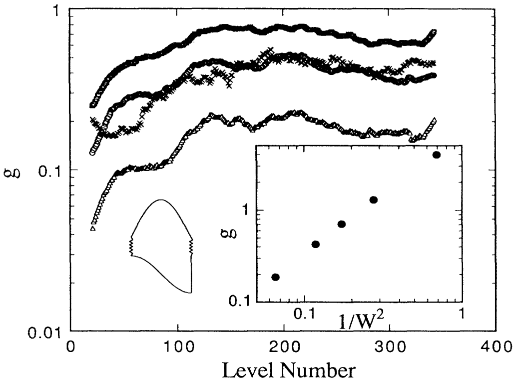

FIG. 2. Variation of g through the spectrum taken from the numerical simulation of an Anderson model on a 27×27 lattice with disorder, $W = 2.4$ (squares), $W = 2.9$ (circles), $W = 3.9$ (triangles), and a chaotic billiard (crosses) with geometry shown inset. The cylindrical geometry of the billiard is constructed by connecting the figure along the zigzag edge. In all cases, the measurements are shown after averaging over forty neighboring levels, and in the case of the disorder, averaging over four realizations of the potential, and combining the data from the upper and lower halves of the spectrum. The variation of g as a function of W^{-2} is shown inset.

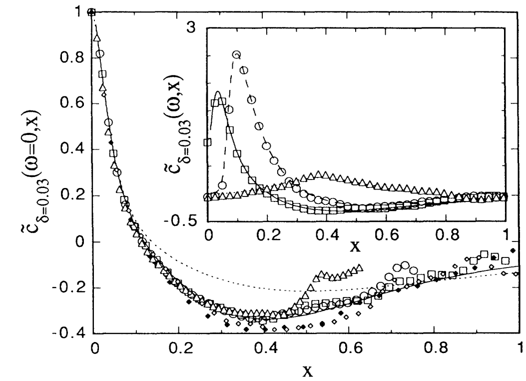

FIG. 3. Autocorrelation of velocities $\tilde{c}_{\delta=0.03}(0,x)$ from unitary theory (full line), orthogonal theory (dotted line) from Ref. [10], numerical simulation of an Anderson model with $X \equiv \phi$ and disorder, $W = 2.4$ (squares), $W = 2.9$ (circles), $W = 3.9$ (triangles), numerical simulation of a chaotic billiard with geometry shown in Fig. 2 (open diamonds); and with $X \equiv BL^2/\phi_0$ and disorder $W = 2.4$ (filled diamonds). We note that the region at which the data begin to deviate from the universal curve increases with increasing g as expected. The variation of $\tilde{c}_{\delta=0.03}(\omega,x)$ for different values of ω from unitary theory (numerical) simulation with disorder $W = 2.4$ is shown inset, with $\omega = 0.1$ [full line (squares)], $\omega = 0.25$ [broken line (circles)], and $\omega = 1$ [dotted line (triangles)]. To enhance the statistics, the disordered samples are each averaged over four realizations of the potential. In all cases, $\tilde{c}_{\delta=0.03}(\omega,x)$ is determined by averaging over a range corresponding to 1/4 of the eigenvalues.

ciding with the value at which an energy level typically changes by an amount ω. Beyond the maximum $\tilde{c}(\omega,x)$ changes sign and approaches the asymptotic behavior of Eq. (11).

We compare the theory with numerical simulation of an Anderson model with on-site energies in the range $-W/2 < W_i < W/2$ (measured in units of the hopping matrix element), and subject to a one-component flux, $X = \phi$. The conductance $g = C(0)/4\pi$ (Fig. 2) shows a W^{-2} dependence consistent with the Born approximation [13].

To compare $\tilde{c}(\omega,x)$ with numerical simulation it is convenient to make a Gaussian regularization of the DOS. This introduces a factor $\exp[-\delta^2\lambda^2]$ into the integrand of Eq. (9) and affects $\tilde{c}(\omega,x)$ only at small x and ω, in particular broadening the δ function at the origin. The agreement of theory with experiment (shown in Fig. 3 for $\delta = 0.03$) *without any free parameters* is striking. A simulation with magnetic field $X \equiv BL^2/\phi_0$ is also in good agreement with theory.

To verify that the universality extends beyond disordered systems we compare measurements of $\tilde{c}(\omega,x)$ taken from a typical chaotic billiard (with geometry shown in Fig. 2). The results shown in Fig. 3 give good agreement with theory, and further demonstrate that spectral and ensemble averaging have the same effect verifying ergodicity.

For complex crystalline lattices, it is convenient to reexpress the rescaling through the mean-square velocity $\bar{v}^2 \equiv C(0)\Delta^2 L^2/4\pi^2$, with $\mathbf{x} \equiv \bar{v}\boldsymbol{\pi}/\Delta$. Then the universal autocorrelation of Fermi velocities can be written as $\bar{v}^2\tilde{c}^{\mu\eta}(\omega,\mathbf{x})$. Approximations used in deriving $k(\omega,\mathbf{x})$ restrict the range of momenta difference to the Brillouin zone. We note that the appropriate symmetry is unitary since the effective Hamiltonian for the periodic part of the Bloch function violates T invariance except when the total momentum is zero.

Having demonstrated that the expression for $k^u(\omega,\mathbf{x})$ leads to an accurate description of two types of chaotic system we will present an outline of its derivation. Representing $k(\Omega,\boldsymbol{\phi})$ in terms of a supersymmetric Lagrangian containing equal numbers of commuting and anticommuting fields, the interaction generated by the ensemble averaging is decoupled by a Hubbard-Stratonovitch transformation with the introduction of 8×8 supermatrix fields, $Q(\mathbf{r})$. To order $(mv^2\tau)^{-1}$ and $(\Delta\tau)^{-1}$, small fluctuations about the saddle point are controlled by a functional nonlinear σ model [10],

$$k(\Omega,\boldsymbol{\phi}) = -\frac{1}{2}\Big[\int Q_{11}^{11}(\mathbf{r})\; Q_{11}^{22}(\mathbf{r}')\exp(-F[Q])\;\mathcal{D}Q\; L^{-2d}\,d\mathbf{r}\,d\mathbf{r}' + 1\Big], \tag{12}$$

where $F[Q]$ denotes the free energy,

$$F[Q]=\frac{\pi}{8L^d\Delta}\int \mathrm{STr}\Big[D\left(\nabla Q-\frac{ih}{L}[Q,\mathbf{\Phi}\tau_3]\right)^2+2i\Omega\Lambda Q\Big]d\mathbf{r}, \tag{13}$$

with the constraint $Q^2=1$. $\mathbf{\Phi}^{\tau\tau'}$ represents a diagonal matrix which depends only on the upper indices with signature $(\bar{\boldsymbol{\phi}},\bar{\boldsymbol{\phi}}+\boldsymbol{\phi})$. For brevity we refer to Ref. [9] for the symmetry properties of Q together with the definition of τ_3, Λ and the supertrace, denoted STr.

For vanishing $\bar{\mathbf{\Phi}}$ and Ω the transverse fluctuations of all the fields in Q become Goldstone modes. The corresponding symmetry of Q belongs to the orthogonal ensemble and both diffuson and Cooperon modes of diffusion contribute significantly to the average. For $\bar{\boldsymbol{\phi}}\neq 0$ and $\boldsymbol{\phi}=0$, the matrix τ_3 breaks T-reversal symmetry and the Goldstone modes corresponding to the Cooperon degrees of freedom acquire a mass. The remaining degrees of freedom belong to the unitary ensemble and are sensitive only to the flux difference $\boldsymbol{\phi}$. Since $\bar{\boldsymbol{\phi}}$ is typically large, we will examine fluctuations of $Q(\mathbf{r})$ which display only unitary symmetry.

Within the diffusive regime, for $|\phi_\mu|<1/2$ the contribution from the nonzero space harmonics can be neglected [9], making the functional integrals that enter Eq. (12) definite. At values of flux $|\phi_\mu|\sim 1/2$ gauge invariance requires a more careful definition of the zero mode [10] from which periodicity can be recovered. However, since this crossover is described by nonuniversal corrections from higher modes, we will henceforth assume that $|\phi_\mu|<1/2$. A treatment of the higher modes by perturbation theory is presented in Ref. [8]. Performing the definite integrations we obtain the expression for $k^u(\Omega,\boldsymbol{\phi})$ given by Eq. (6).

A perturbation which conserves T invariance can be used to determine the universal function for the orthogonal ensemble. The procedure [10] is analogous to that presented above and gives

$$k^o(\omega,x)=\mathrm{Re}\int_{-1}^{1}d\lambda\int_1^\infty d\lambda_1\int_1^\infty d\lambda_2\frac{(1-\lambda^2)(\lambda-\lambda_1\lambda_2)^2}{(2\lambda\lambda_1\lambda_2-\lambda^2-\lambda_1^2-\lambda_2^2+1)^2}\times\exp[-\pi^2\mathbf{x}^2(2\lambda_1^2\lambda_2^2-\lambda_1^2-\lambda_2^2-\lambda^2+1)/4-i\pi\omega(\lambda-\lambda_1\lambda_2+i\delta)]. \tag{14}$$

The expression differs qualitatively from the unitary case by the additional integral over λ_2. This is a remnant of the Cooperonic degrees of freedom which give a nonvanishing contribution when the symmetry is orthogonal.

In conclusion, we have studied the dispersion of the energy levels of quantum chaotic systems in response to an Aharonov-Bohm flux. The functional form of the autocorrelator of DOS fluctuations suggest a natural rescaling in terms of the mean-level spacing and $C(0)$ in which the dependence on detailed properties of the system is removed. The analytical results have been verified by numerical simulation of models with scattering from impurities (disorder), as well as from irregular boundaries (billiard), the latter suggesting that $k^u(\omega,x)$ is not specific to disorder but describes a wider class of chaotic systems. In addition to $k^u(\omega,x)$, we have presented the universal function for the orthogonal ensemble, $k^o(\omega,x)$ [10]. We suggest that the universality after rescaling applies to *all statistical properties of the random functions* $\epsilon_i(x)$, and with the same generality as the Wigner-Dyson distribution, providing a new characterization of quantum chaotic systems.

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