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Super Lax Pairs and Infinite Symmetries in the $1/r^2$ System

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We present an algebraic structure that provides an interesting and novel link between supersymmetry and quantum integrability. This structure underlies two classes of models that are exactly solvable in one dimension and belong to the $1/r^2$ family of interactions. The algebra consists of the commutation between a "super-Hamiltonian" and two other operators, in an enlarged Hilbert space. These reduce to quantal ordered Lax equations when projected onto the original subspace, and to a statement about the "harmonic lattice potential" structure of the Lax operator, leading to a highly automatic proof of the integrability of these models and to an interesting hierarchy of new models.

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In this work we present a novel algebraic structure that has arisen in the course of our recent work on the $1/r^2$ family of many body problems. This structure seems to be of possible interest in several many body systems, and also in the context of field-theoretic problems that lead to a study of matrix models of different kinds, including those with fermionic degrees of freedom [1]. We find a new and intimate relationship, within the models considered here, between concepts that are of great interest in their own right, namely, supersymmetry [2] and quantum integrability. We find, remarkably enough, that the quantum integrability of certain bosonic quantum systems is easier to understand by enlarging the Hilbert space, and embedding these in problems containing fermionic degrees of freedom as well. Under this enlargement, certain nontrivial Lax relations for the bosonic systems turn into simple commutators in the enlarged system. The Lax equations, unlike in the classical case [3–6], do not imply integrability in the quantum models in general due to severe quantum ordering problems. In the models considered here, however, integrability follows from the structure of our equations in a highly automatic fashion. From the point of view of supersymmetry, nontrivial models exhibiting this symmetry are shown to be made up of more fundamental operators, which are invisible at the usual level of description.

Our examples in this work are the continuum Sutherland-Calogero-Moser (SCM) $1/r^2$ [3, 7] system, and the discrete SU(n) symmetric $1/r^2$ system solved by Shastry [8] and independently by Haldane [9]. These and derived models have been very popular, and much progress has been made regarding the details of the solutions. However, a deeper understanding of the structure of the solutions has been less easy to obtain. We present here a unifying idea, an algebraic structure that seems to capture the integrability of this class of problems. Such relations are of interest because they are "irreducible" statements about the family of models. It is therefore crucial to ask for a "structural" content of solvable models, and our paper throws light in this direction for this class of models.

We first present the algebraic structure, and show how quantum integrability follows very simply from this. We first discuss the SCM system where the algebraic structure is realized, and also where supersymmetry in the usual sense [2, 10, 11] is fulfilled. We next study the discrete SU(n) : $1/r^2$ model, this system has received considerable attention very recently from diverse points of view [11–18]. In this work, we show that this model fits very naturally into the above structure. Morover, our scheme uncovers a fascinating hierarchical structure of Hamiltonians, wherein at each stage the larger Hilbert

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space "super-Hamiltonian" has again the form of the $SU(n) : 1/r^2$ model, but with a different number of components, some necessarily fermionic.

General algebraic structure.—We begin by highlighting the general algebraic structure that emerges from the detailed models, requiring a triad of operators, μ, $\mathcal{H}$, and $\mathcal{L}$ with specific commutation relations. We consider a system of N bosonic degrees of freedom [19], say N bosons or N sites in a spin chain, and append to this set N fermionic degrees of freedom, i.e., operators c_n, $c_m^\dagger$ obeying canonical anticommutation rules. The "uniform mode" operator $\mu = \sum_{n=1}^{N} c_n$, an unnormalized Fermi operator ($\mu^2 = 0$), is the first member of the triad. We next require a Hermitian super-Hamiltonian $\mathcal{H}$ of the form

$$\mathcal{H} = H_b + H_f, \tag{1}$$

with H_b consisting purely of bosonic variables and $H_f = \sum_{i,j} M_{i,j} c_i^\dagger c_j$, with $M_{i,j}$ purely bosonic. The third operator is the super Lax operator $\mathcal{L}$ in the form

$$\mathcal{L} = \sum_{i,j} L_{i,j} c_i^\dagger c_j, \tag{2}$$

where again $L_{i,j}$ is bosonic and obeys $L_{i,j}^\dagger = L_{j,i}$ so that $\mathcal{L}$ is Hermitian. The two fundamental commutations we require are

$$[\mathcal{H}, \mathcal{L}] = [\mathcal{H}, \mu] = 0. \tag{3}$$

The commutation with μ requires a constraint on the form of M, namely, $\sum_i M_{i,j} = 0 = \sum_j M_{i,j}$. The super Lax commutation relation requires a highly nontrivial constraint on the functional forms of the operators L, M, and H_b. The operators $\mathcal{L}$ and μ do not commute with each other in general. It follows from Eq. (3) that any operator function of μ and $\mathcal{L}$, say $f(\mu, \mathcal{L})$, commutes with $\mathcal{H}$. This scheme provides us with an elegant formulation of the integrability of the purely bosonic model H_b. We introduce the notion of bosonic projection of an arbitrary "super" operator: $f \longrightarrow \hat{f}_{\alpha,\beta} = \langle 0|\mu^\alpha f (\mu^\dagger)^\beta|0\rangle$, where $|0\rangle$ is the fermionic vacuum state defined by $c_n|0\rangle = 0$, with $\alpha, \beta = 0, 1$ giving four possible results. It follows that

$$\{[\mathcal{H}, f] = 0\} \Rightarrow [H_b, \hat{f}_{\alpha,\beta}] = 0. \tag{4}$$

To see this, take the matrix element of $[\mathcal{H}, f] = 0$ in $(\mu^\dagger)^\beta|0\rangle$ and $(\mu^\dagger)^\alpha|0\rangle$, giving $[H_b, \hat{f}_{\alpha,\beta}] = \langle 0|(\mu)^\alpha [f, H_f](\mu^\dagger)^\beta|0\rangle$; however, Eq. (3) allows us to commute H_f past μ and annihilate $|0\rangle$ proving Eq. (4). In practice, only one of the four projections is of use in giving nontrivial results. The scheme for constructing an infinite number of commuting operators is clear; we take various functions f and project them. Thus the integrability of H_b is closely connected to the above embedding into our algebra Eq. (3). Specific examples of functions are easy to find; one obvious choice is to let $f = L^n$, where n is an integer. In this case the useful projection is $(\hat{\mathcal{L}^n})_{1,1} = \sum L_{i_1,i_2} L_{i_2,i_3} \cdots L_{i_n,i_{n+1}}$, which may be written simply as $\mathrm{Tr}(L^n \Lambda)$ viewing L as a $N \times N$ matrix and the special matrix $\Lambda_{i,j} = 1 \ \ \forall\, i, j$. Another interesting class of functions is $\mathcal{G}(t) = \{\mu(t), \mu(0)^\dagger\}$, where $\mu(t) = \exp(it)\mu \exp(-it)$ and t is a spectral parameter. Expanding in t we find $\mu(t)$ contains $1, 3, 5, \ldots$ Fermi operators and hence $\mathcal{G}(t)$ contains even Fermi operators. Explicit examples are detailed below.

The first of the fundamental relations, Eq. (3), is in fact one in the sense of Lax and Moser [3]. In the examples considered in this work, the commutator $[\mathcal{H}, \mathcal{L}]$ reduces, remarkably enough, to a bilinear in fermions with a bosonic coefficient, the vanishing of this coefficient is a quantal ordered Lax (OL) equation

$$[L_{i,j}, H_b] = \sum_k (M_{i,k} L_{k,j} - L_{i,k} M_{k,j}). \tag{5}$$

In this equation L and M are bosonic operators, and so the ordering of the terms in Eq. (5) is crucial. The early work of Calogero, Ragnisco, and Marchioro (CRM) [4] quantized the classical Lax equation written down by Moser [3], by antisymmetrizing the right-hand side (RHS) of Eq. (5) in the quantum sense. The above ordered Lax (OL) equation has a natural matrix product like order built into it leading to a "telescopic" cancellation of internal terms, whereby $[L^n, H_b] = (ML^n - L^n M)$. Further, if the matrix relation $M\Lambda = \Lambda M = 0$, with $\Lambda_{i,j} = 1$, then we see that $\Lambda[L^n, H_b]\Lambda = 0$. The condition $M\Lambda = 0$ is recognizable as the condition $[H_f, \mu] = 0$. We emphasize that in the present treatment the OL equation (5) does not lead to integrability, unlike the classical Lax case, and we further need the condition $M\Lambda = 0$.

Continuum $1/r^2$ model.—We begin with the SCM model [7], where ($x_i \in [0, L]$),

$$H_b = \sum_i p_i^2 + 2\lambda(\lambda - 1)\phi^2 \sum_{i<j} \sin^{-2}[\phi(x_i - x_j)] - E_0, \tag{6}$$

with $\phi = \pi/L$ and $E_0 = N(N^2 - 1)\phi^2\lambda^2/3$, and write for the Lax operator

$$L_{i,j} = p_i\, \delta_{i,j} + \chi_{i,j}\, (1 - \delta_{i,j}), \tag{7}$$

where the function χ is to be determined. The commutator $[\mathcal{H}, \mathcal{L}]$ would be trivially bilinear in the c's if the L and M commuted independently of the statistics of the variables' c's. As it is, we have all the off diagonal elements of L commuting with M's (assumed to be functions of x_i), and only $L_{i,i} = p_i$ does not commute with M's. We note, however, that in the case of $L_{i,i}$ the four-fermion operator generated is $c_i^\dagger c_i c_i^\dagger c_k$, which reduces to $c_i^\dagger c_k$ if we make the choice that c's are fermions. This choice of the statistics of c's enables the commutator to be reduced entirely to a bilinear in the fermions:

$$[\mathcal{L}, \mathcal{H}] = \sum_{i,j} c_i^\dagger c_j \left\{ [L_{i,j}, H_b] + \sum_k (L_{i,k} M_{k,j} - M_{i,k} L_{k,j}) \right\}. \tag{8}$$

The condition for its vanishing is just the OL equation (5). The functions χ and M can be found by writing $M_{i,j} = \delta_{i,j}\sum_k d_{i,k} + (1-\delta_{i,j})m_{i,j}$, and writing the interaction more generally in Eq. (6) as $1/2\sum v_{i,j}$, giving functional relations $m_{i,j} = 2\chi'_{i,j}$, $v_{i,k} = (\text{const} - d_{i,k} + 2\chi^2_{i,k})$, and $\chi_{i,j}(d_{i,k} - d_{j,k}) = \chi_{i,k}m_{k,j} - m_{i,k}\chi_{k,j}$. This set of equations has a large class of solutions, with elliptic functions for $v_{i,j}$, being the most general answer. However, in view of Eq. (3), we demand $d_{i,k} = -m_{i,k}$, which cuts down on the allowed solutions drastically. The essentially unique solution (with periodic boundaries) is

$$\chi_{i,j} = \phi\lambda\cot[\phi(x_i - x_j)], \tag{9}$$

$$M_{i,j} = 2\lambda\phi^2\Big[\delta_{i,j}\sum_k{}' \sin^{-2}\phi(x_i - x_k) - (1-\delta_{i,j})\sin^{-2}\phi(x_i - x_j)\Big]. \tag{10}$$

Another, essentially equivalent solution is found by replacing $\lambda \longrightarrow (1-\lambda)$ in the above Eq. (10). Note that $\sum_i M_{i,j} = 0$ in this case by the "harmonic lattice potential" structure of the operator M leading to the matrix equation $M\Lambda = 0 = \Lambda M$. We remark that the structure of M and L is identical to that found in the classical case by Moser, and also by CRM.

In this model we have also supersymmetry in the usual sense [2] ($n = 1$ complex supersymmetry): we define a new Fermi-like operator

$$\zeta = \sum Q_i c_i^\dagger, \quad Q_i \equiv \sum_j L_{i,j} \tag{11}$$

with $\zeta = [\mathcal{L}, \mu^\dagger]$. The operators $[Q_i, Q_j] = 0$, whereby $\zeta^2 = 0$. In fact the ground state of Eq. (6) is annihilated by each of the Q_i, as noted in Ref. [11]. We can use the standard supersymmetry construction and form the operator

$$H_{\rm ss} = \{\zeta, \zeta^\dagger\}. \tag{12}$$

By construction we have the property $[H_{\rm ss}, \zeta] = 0$. Supersymmetry of the system in this sense was already noted in [10, 11]. It is easy to show by explicit calculation for this model that $H_{\rm ss} = \mathcal{H}$. Note that *all* the operators in this supersymmetric theory are generated by two underlying fundamental operators $\mathcal{L}$ and μ.

We remark that the entire structure presented here goes through for the case of open boundaries, i.e., the Calogero system. The results may be obtained by similar calculations, or most simply by making replacements in the operators: $[\phi\cot(\phi x)] \to 1/x$ and $[\phi/\sin(\phi x)] \to 1/x$. We also note that the explicit form of the operators M, H_b enables us to write the super-Hamiltonian

$$\mathcal{H} = \sum_i p_i^2 + 2\lambda\sum_{i<j}\sin^{-2}[\phi(x_i - x_j)][\lambda - \{1 - (c_i^\dagger - c_j^\dagger)(c_i - c_j)\}] - E_0. \tag{13}$$

The term in the curly brackets in Eq. (13) has the form of a permutation operator and is discussed in the next section.

Discrete $SU(n)$: $1/r^2$ model.—Here we consider the discrete $1/r^2$ model with

$$H_b = \sum_{i<j} v_{i,j}P_{i,j}, \tag{14}$$

where $v_{i,j} = v_{j,i}$ is the interaction. The exact solution [8, 9] was given for the case $v_{i,j} = \phi^2/\sin^2[\phi(x_i - x_j)]$, but we shall keep its form general until later. Further the x_i are on a uniform lattice $x_i \in Z_N$. Here $\phi = \pi/N$ and $P_{i,j}$ the permutation operator in SU(n) [it is $\frac{1}{2}(1 + \vec{\sigma_i}\cdot\vec{\sigma_j})$ for spin 1/2 hard core bosons corresponding to $n = 2$). In this model we make the choice for the Lax operator

$$L_{i,j} = (1-\delta_{i,j})\, l_{i,j}\,(P_{i,j} + c), \tag{15}$$

where $l_{i,j}$ and c are undetermined as yet, and for the operator M we assume

$$M_{i,j} = \delta_{i,j}\sum_k{}' g_{i,k}(P_{i,k} + d) - (1-\delta_{i,j})g_{i,j}(P_{i,j} + d), \tag{16}$$

where d and the function $g_{i,k}$ are as yet undetermined. Carrying out the commutator in Eq. (3), we find

$$[\mathcal{L}, \mathcal{H}] = \sum_{i,j} c_i^\dagger c_j\Big([L_{i,j}, H_b] + \sum_k (L_{i,k}M_{k,j} - L_{k,j}M_{i,k})\Big) + \sum_{i,j,k,l}[L_{i,j}, M_{k,l}]c_k^\dagger c_l c_i^\dagger c_j. \tag{17}$$

The last term generates a four-fermion term with distinct i, j, k,

$$\sum_{i,j,k}{}' c_k^\dagger c_k c_i^\dagger c_j [P_{i,j}, P_{i,k}] \times (l_{i,j}\{g_{k,j} - g_{k,i}\} + l_{i,k}g_{k,j} - l_{k,j}g_{i,k}),$$

which is required to vanish, yielding a functional constraint (for unequal i, j, k)

$$(l_{i,j} - l_{i,k})g_{k,j} = (l_{i,j} - l_{k,j})g_{i,k}. \tag{18}$$

The third term in the RHS of Eq. (17) becomes $c_i^\dagger c_j g_{i,k}(l_{i,j} + l_{k,j})[P_{i,j}, P_{i,k}]$, and after some rearrangement, becomes identical to Eq. (8). The vanishing of the quadratic form yields again the OL equation (5). The OL equation can be worked out readily. Omitting the constants c, d we see that the OL equation becomes

$$\sum_k{}'\{P_{i,j}P_{i,k}[l_{i,j}(v_{i,k} - v_{j,k}) + g_{i,k}l_{k,j} - l_{i,k}g_{k,j}] + P_{i,k}P_{i,j}(v_{j,k} - v_{i,k} + g_{j,k} - g_{i,k})l_{i,j}\} = 0. \tag{19}$$

This yields $g_{i,j} = g_{j,i} = -v_{i,j}$, and further the same constraint as Eq. (18) results. This equation can be written as $l(x+y) = [l(x)v(y) - l(y)v(x)]/[v(y) - v(x)]$. It is clear that if $l(x)$ is a solution then so is $l + \text{const}$. Nontrivial solutions can be found using the evenness of v, by writing $y = -x + \epsilon$, and expanding $v(y)$ around $-x$. To leading order in ϵ, we find $l(\epsilon) = [1/\epsilon][l(x) - l(-x)]v(x)/v'(x)$, which implies that nontrivial solutions in l are odd functions, and further that $l(x) \propto v'(x)/v(x)$. We thus end up with a functional equation satisfied by v alone. With periodic boundaries, this has a unique solution $v \propto 1/\sin^2(x)$. This also determines the function $l_{i,j} = ip\cot[\phi(x_i - x_j)] + q$, where p, q are real constants. The constants in Eq. (16) can be found after a tedious calculation as $d = \frac{2}{3}c$.

The super-Hamiltonian $\mathcal{H}$ can be written explicitly using the above solution (with $c = 0$) as

$$\mathcal{H} = \sum_{i<j} v_{i,j} P_{i,j} \Xi_{i,j}, \tag{20}$$

where $\Xi_{i,j} = 1 - (c_i^\dagger - c_j^\dagger)(c_i - c_j)$. It is clear that under a particle-hole transformation $c_i \to c_i^\dagger$, the spectrum of $\mathcal{H}$ is inverted, and in the "half filled" sector $N_F = N/2$, the spectrum has inversion symmetry about zero. Moreover, if $|\Psi\rangle$ is an eigenfunction of $\mathcal{H}$ and $\hat{N_F}$, with eigenvalues E and N_F, then $\mu|\Psi\rangle$ is either null, or is degenerate with $|\Psi\rangle$, with $N_F - 1$ fermions.

It is a remarkable and unexpected fact that $\Xi_{i,j}$ also satisfies the generator relations for the permutation group $\Pi_{i,j}^2 = 1$ and $\Pi_{i,j}\Pi_{j,k}\Pi_{i,j} = \Pi_{j,k}\Pi_{i,j}\Pi_{j,k}$, and so does the composite operator ΞP. We thus see that the super-Hamiltonian is again an SU(m) : $1/r^2$ model. The difference of course, is that the added fermions increase the number of components of the model. "Grading" of this sort was discussed for the Yang-Baxter integrable family of models in [20], wherein the so-called "R-matrix" of the graded SU($m+n$) case is well known [21].

More explicitly, let the starting "bosonic" model [19] consist of n_b bosonic species and n_f fermionic species, so $n = n_b + n_f$. We always restrict ourselves to states in the Hilbert space where we have only one particle at each site. By adding one new set of fermions to this model, we are forcing the fermions to sit on top of one of the other particles. Once the new fermions are "glued" to preexisting particles, they hop along with the carrier particles by the action of the Hamiltonian, and hence we are effectively increasing the number of species. Clearly gluing a fermion (boson) to a boson or a fermion produces a fermion (boson) or a boson (fermion) and hence the super-Hamiltonian is an SU(n') : $1/r^2$ system with $n' = 2n$ and with $n_f' = n_f + n_b$ and with $n_b' = n_f + n_b$. The principal series of models then is of the kind SU($2^{\nu+1}$), with 2^ν kinds of bosons and an equal number of Fermi species ($\nu = 0, 1, \ldots$). At the level ν the super-Hamiltonian is related as in our algebraic scheme to several different (lower n) models, in fact $2^\nu + 1$ of them. It should be clear that the model with added fermions has a spectrum that is highly reducible, in the sense that it has several constants of motion, namely, the number of distinct species, with regard to which the Hamiltonian breaks up into blocks.

We have examined the eigenspectrum of the $\mathcal{H}$ for small systems numerically; it shows fascinating degeneracies and a rich structure, and we will return to its discussion in a later work.

In conclusion, the algebraic structure in Eq. (3) reported in this work provides a novel framework for understanding the integrability of the $1/r^2$ family of problems. The connection between supersymmetry and integrability arises through an underlying Lax equation, and this relationship is clarified in the present set of problems. We see that the statement of having an infinite number of conservation laws for the bosonic systems, is in fact no more than the existence of *two* constants of motion of a certain kind in the enlarged problem. This remarkable "contraction" is made possible by the enlargement of the Hilbert space. Our framework further provides a natural way to understand the asymmetric structure of the ubiquitous (quantum bosonic) Lax equation, equating a quantum commutator (or a Poisson bracket in the classical case) to a matrix commutator; we show that this relation is the projection of a simple commutator in the enlarged Hilbert space. Our identification of the hierarchical structure of the "super Hamiltonians" presents interesting and new problems that are of considerable interest.

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