Solvable Spin Glass of Quantum Rotors

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We examine a model of M-component quantum rotors coupled by Gaussian-distributed random, infinite-range exchange interactions. A complete solution is obtained at $M = \infty$ in the spin-glass and quantum-disordered phases. The quantum phase transition separating them is found to possess logarithmic violations of scaling, with no further modifications to the leading critical behavior at any order in $1/M$; this suggests that the critical properties of the transverse-field Ising model (believed to be identical to the $M \to 1$ limit) are the same as those of the $M = \infty$ quantum rotors.

Extensive attention has been lavished in the last decade on the finite-temperature properties of classical and quantum spin glasses [1–5]. In contrast, there has been almost no work on the $T = 0$ properties of quantum spin glasses, and, in particular, on the quantum phase transition from a spin-glass to a quantum-disordered (or "spin-fluid") state. The critical properties of this quantum phase transition have so far been obtained only for a one-dimensional model [6] which is unfrustrated in the classical limit. On the experimental side, there has been a renewed interest in a number of spin systems which are in the vicinity of a $T = 0$ phase transition from a spin-glass to a spin-fluid state [7–11]: these include the dipolar, transverse-field Ising magnet LiHo$_x$Y$_{1-x}$F$_4$ [7], the lightly doped cuprates [8–10], and various layered transition-metal/rare-earth oxides [11].

In this paper we examine a quantum spin glass which allows us to examine more carefully the nature of the quantum spin-glass to spin-fluid phase transition and determine the spectrum of excitations in the spin-fluid phase. We consider M-component quantum rotors with Gaussian-distributed random, infinite-range exchange interactions. A complete solution of this model will be obtained at $M = \infty$ in both the spin-glass and spin-fluid phases and at the critical point separating them. We also examine the nature of the $1/M$ corrections at $T = 0$ in the spin-fluid phase and at the critical point: we find that the form of the leading critical behavior and the low-frequency spectral weight remains unmodified to all orders in $1/M$ from the $M = \infty$ result. Thus the results of this paper could have been derived without any reference to the $1/M$ expansion, by simply resumming Feynman graphs which are dominant at low frequency —these graphs happen to be identical to those selected by the $M = \infty$ theory.

The quantum rotors should not be confused with true quantum Heisenberg spins present in any isotropic antiferromagnet; the different components of the rotor variables all commute with each other, unlike the quantum spins. As a consequence, the path integral written in the rotor variables has an action which contains no Berry phases and is purely real. The properties of random quantum spin models are quite different from those of the quantum rotors considered here, and will be discussed elsewhere [12]. Apart from its theoretical simplicity, the main utility of the rotor model is that the $M = 1$ limit of the path integral is expected to be in the same universality class as the Ising model in a transverse field. The absence of any $1/M$ corrections, noted above, suggests that the critical behavior of the infinite-range, transverse-field Ising model is identical to that of the $M = \infty$ limit solved in this paper. This is also consistent with a recent analysis of this Ising model by Huse and Miller [13]: their results for the critical point are essentially identical to those obtained below in the $M = \infty$ model.

We will study the following ensemble of Hamiltonians:

$$H = \frac{g}{2M} \sum_i \hat{L}_i^2 + \frac{M}{\sqrt{N}} \sum_{i<j} J_{ij} \hat{n}_i \cdot \hat{n}_j, \qquad \hat{n}_i^2 = 1, \quad (1)$$

where i, j extend over N sites, $n_{i\mu}$ are the M components of a unit-length rotor $\hat{n}_i$ on site i, the $L_{i\mu\nu}$ ($\mu < \nu$, $\mu, \nu = 1, \ldots, M$) are the $M(M-1)/2$ components of the angular-momentum generator $\hat{L}_i$ in rotor space, and the J_{ij} are mutually uncorrelated exchange constants selected with probability $P(J_{ij}) \sim \exp(-J_{ij}^2/2J^2)$. The $n_{i\mu}$ are mutually commuting variables and the quantum dynamics is defined by the commutation relations

$$[L_{i\mu\nu}, n_{j\sigma}] = i\delta_{ij}(\delta_{\mu\sigma}n_{j\nu} - \delta_{\nu\sigma}n_{j\mu}). \quad (2)$$

The $L_{i\mu\nu}$ satisfy the commutation relations of angular momenta in M dimensions. As $g \to 0$, the model reduces to the *classical*, infinite-range, M-component, Heisenberg spin glass which was analyzed earlier by de Almeida *et al.* [14].

The formulation of the $N \to \infty$ limit of H can be obtained by a straightforward generalization of the analyses in Refs. [2–4]. We use the path-integral formulation of the partition function, introduce n replicas, and average over the ensemble of the J_{ij}. The $N \to \infty$ limit yields a saddle point which describes the quantum mechanics of n replicas of a single rotor. Assuming the saddle point is O(M) invariant (this is true in both the spin-fluid and spin-glass phases) we obtain the single-site path integral

$$Z_0 = \int \mathcal{D}\hat{n}^a(\tau)\delta(\hat{n}^{a2}(\tau) - 1)\exp\left(-\frac{M}{2g}\int_0^\beta d\tau(\partial_\tau \hat{n}^a)^2 + \frac{MJ^2}{2}\int_0^\beta d\tau d\tau' Q^{ab}(\tau - \tau')\hat{n}^a(\tau)\cdot\hat{n}^b(\tau')\right) \tag{3}$$

and the self-consistency condition

$$Q^{ab}(\tau - \tau') = \langle\hat{n}^a(\tau)\cdot\hat{n}^b(\tau')\rangle_{Z_0}. \tag{4}$$

Here $a, b = 1, \ldots, n$ are replica indices, τ, τ' are Matsubara times, and $\beta = 1/T$. The Edwards-Anderson order parameter [1] for the spin-glass phase is

$$q_{EA} = Q^{aa}(\tau \to \infty). \tag{5}$$

Moreover, Q^{ab}, $a \neq b$, is τ independent and nonzero only in the spin-glass phase [4].

An exact evaluation of Z_0 is clearly not possible. We present below the results of a systematic $1/M$ expansion on Z_0.

$M = \infty$ *theory.*—Imposing the constraint by a Lagrange multiplier λ, the $M = \infty$ limit of Eqs. (3) and (4) reduces to the constraint $Q^{aa}(\tau = 0) = 1$ and

$$Q(i\omega_n) = g\left[\omega_n^2 + \lambda - gJ^2Q(i\omega_n)\right]^{-1}, \tag{6}$$

where $Q(i\omega_n)$ is the Fourier transform of $Q(\tau)$ at the Matsubara frequencies, and the right-hand side is a matrix inverse in replica space.

(1) Paramagnetic phase: For large g, or large T, we expect a paramagnetic phase (the quantum-disordered phase is the $T = 0$ paramagnetic state) in which case Q^{ab} will be replica diagonal [3,4]. A closed-form solution can be obtained from (6) for the spectral weight $\chi''(\omega) = \text{Im}[Q^{aa}(\omega + i0^+)]$:

$$\chi''(\omega) = \text{sgn}(\omega)\frac{[(\omega^2 - \lambda + 2Jg)(\lambda + 2Jg - \omega^2)]^{1/2}}{2J^2g} \tag{7}$$

for $\lambda - 2Jg < \omega^2 < \lambda + 2Jg$ and $\chi'' = 0$ otherwise. It is clear that a physically sensible solution requires $\lambda \geq 2Jg$ where λ is determined by the constraint equation $\hat{n}^{a2} = 1$, or

$$\int_0^\infty \frac{d\omega}{\pi}\chi''(\omega)\coth(\beta\omega/2) = 1. \tag{8}$$

It is evident from (7) that the $M = \infty$ paramagnet has a gap of $(\lambda - 2Jg)^{1/2}$ towards spin-wave excitations. We expect $1/M$ fluctuations to fill in this gap at any finite T; the gap in the $T = 0$ spin-fluid phase, however, is robust towards such corrections. The paramagnetic–spin-glass phase boundary is determined by setting $\lambda = 2Jg$ and solving (8) for a line in the g-T plane: the results of this calculation are shown in Fig. 1. The quantum transition near $T = 0$ occurs at $g = 9\pi^2J/16 - 3T^2/J + \cdots$, and the classical transition near $g = 0$ occurs at $T = J - g/12 + \cdots$; this latter result agrees with that of Ref. [14].

(2) Spin-glass phase: We now expect *only* $Q^{ab}(i\omega_n = 0)$ to acquire off-diagonal components [3,4]; the finite frequency $Q(i\omega_n)$ remains diagonal. We therefore

parametrize

$$Q^{aa}(i\omega_n) = Q^{aa}_{reg}(i\omega_n) + \beta q_{EA}\delta_{\omega_n,0}, \tag{9}$$

where $Q^{aa}_{reg}(i\omega_n)$ can be obtained immediately from the solution of (6) and continues to have spectral weight $\chi''_{reg}(\omega)$ which obeys (7) with a value of λ to be determined below. We parametrize the off-diagonal components of $Q^{ab}(i\omega_n = 0)$ by an arbitrary hierarchical matrix [15] specified by a monotonic function $\beta q(x)$ on the interval $0 \leq x \leq 1$. Using the expressions for the inverse of a hierarchical matrix in Ref. [16], the self-consistency equation (6) can be transformed into two integral equations for $q(x)$ and q_{EA}. Simple algebraic manipulations then yield the satisfactory [1] result

$$q_{EA} = q(1). \tag{10}$$

Repeated differentiation of the integral equations showed that $dq/dx = 0$; $q(x)$ can therefore only be a piecewise constant function. We chose $q(x) = q_1$ for $0 < x < u$ and $q(x) = q_{EA}$ for $u < x < 1$, whence the integral equations specified $q_1 = 0$ and q_{EA}; u, however, was left undetermined [16]. It was then necessary to evaluate the free energy and demand stationarity with respect to u. The final result was quite simple: we found $u = 0$ implying that $q(x) = q_{EA}$ for all x and that the replica-symmetric solution is optimal. This agrees with the classical limit at $g = 0$ which was found in Ref. [14] to possess a stable replica-symmetric solution at $M = \infty$; we also undertook a stability analysis, similar to that in Ref. [14], for the quantum-rotor model and found only non-negative eigenvalues in the fluctuations about the replica-symmetric
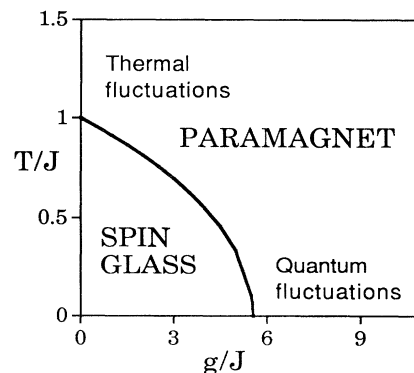

FIG. 1. Phase diagram of H in the T-g plane at $M = \infty$. The line $g = 0$ corresponds to the classical model of Ref. [14]. The quantum-disordered phase is the paramagnet at $T = 0$. Regions in which the spin fluctuations are primarily thermal or quantum are noted.

state. Our final results for the spin-glass phase were $\lambda = 2Jg$ with $\chi''_{\text{reg}}(\omega)$ given by (7) being gapless over the entire phase, $Q^{ab}(i\omega_n = 0) = \beta q_{\text{EA}}$ for $a \neq b$, and

$$q_{\text{EA}} = 1 - \int_0^\infty \frac{d\omega}{\pi} \chi''_{\text{reg}}(\omega)\coth(\beta\omega/2). \qquad (11)$$

(3) Quantum critical region: We now examine the region near the quantum phase transition at $g = g_c \equiv 9\pi^2 J/16$, $T = 0$. Scaling (see, e.g., Ref. [10]) predicts that the spin-glass paramagnetic boundary obeys $T \sim |\delta g|^{z\nu}$ (here $\delta g \equiv g - g_c$). From the equation for the phase-boundary at small T above, we deduce $z\nu = 1/2$. The order parameter q_{EA} must vanish as $q_{\text{EA}} \sim |\delta g|^\beta$; from (11) this yields $\beta = 1$. Further the $T = 0$ spin gap, Δ, in the quantum-disordered phase should vanish as $\Delta \sim (\delta g)^{z\nu}$. Using $\Delta = (\lambda - 2Jg)^{1/2}$ and (8), we find, however, that $\Delta \sim [\delta g/\ln(1/\delta g)]^{1/2}$. Thus there is a surprising logarithmic violation of naive scaling—the logarithmic divergence is a consequence of the square-root threshold in the spectral weight (7). For ω and δg small ($\omega, \delta g \ll g, J$), but $\omega/\delta g$ arbitrary, the entire $T = 0$, local dynamic susceptibility obeys a scaling form

$$\chi''(\omega, T = 0) = c_1 \text{sgn}(\omega)|\omega|^\mu \Phi_g\left(\frac{\omega}{\Delta_g}\right), \qquad (12)$$

where the frequency scale Δ_g obeys $\Delta_g = c_2(\delta g)^{z\nu}/\log^{1/2}(1/\delta g)$ for small δg, the exponent $\mu = -1 + \beta/(z\nu) = 1$ [10], c_1, c_2 are nonuniversal constants, and Φ_g is a universal function given by

$$\Phi_g(x) = \begin{cases} (1 - 1/x^2)^{1/2}, & \text{for } |x| > 1, \\ 0, & \text{otherwise.} \end{cases} \qquad (13)$$

We will argue below that the results for $z\nu$, β, μ, and Φ_g are in fact exact to all orders in $1/M$; only the nonuniversal constants c_1, c_2 get modified by higher-order corrections. A related analysis can be performed at the critical coupling $g = g_c$ but at finite temperature [10]. For ω and T small ($\omega, T \ll g, J$), but with ω/T arbitrary, the local dynamic susceptibility now obeys the scaling form

$$\chi''(\omega, g = g_c) = c_1 \text{sgn}(\omega)|\omega|^\mu \Phi_T\left(\frac{\hbar\omega}{\Delta_T}\right), \qquad (14)$$

where the universal function Φ_T is

$$\Phi_T(x) = \begin{cases} (1 - 4\pi^2/3x^2)^{1/2}, & \text{for } |x| > 2\pi/\sqrt{3}, \\ 0, & \text{otherwise,} \end{cases} \qquad (15)$$

and the frequency scale $\Delta_T = k_B T/\log^{1/2}(1/T)$ at low T, with no nonuniversal prefactor. Note again the presence of logarithmic violations of naive scaling; the frequency scale for the dynamic susceptibility, however, is still set completely by the absolute temperature to leading-log accuracy. Our result for Φ_T is also exact to all orders in $1/M$.

1/M expansion.—We now examine corrections to the above mean field theory at $T = 0$ in the quantum disor-

dered phase and at the quantum critical point, $g = g_c$. We will not examine such corrections in the spin-glass phase where the structure is considerably more complicated due to the expected appearance of replica symmetry breaking. Our main result will be that neither the critical exponents nor the form of the low-frequency spectral weights are modified by the $1/M$ corrections. We begin by absorbing all higher-order corrections into a self-energy, Σ, in the $\hat{n}$ propagator, which modifies (6) to

$$Q(i\omega_n) = g\left[\omega_n^2 + \lambda - gJ^2 Q(i\omega_n) + \Sigma(i\omega_n)/M\right]^{-1}. \qquad (16)$$

The function $\Sigma(\tau)$ is itself a nonlinear functional of $Q(\tau)$, obtainable by a $1/N$ expansion of Z_0. Let us consider first the critical point $g = g_c$ and use the $M = \infty$ result $Q^{aa}(i\omega_n) \sim |\omega_n|$ at low frequencies. The leading term in Σ satisfies $\text{Im}[\Sigma(\omega + i0^+)] \sim \omega^5$, $\text{Re}[\Sigma(\omega + i0^+)] \sim a_1 + a_2\omega^2$, at small ω; the suppression at low frequencies in $\text{Im}(\Sigma)$ arises from restriction in the phase space to three spin-wave decay. On the imaginary frequency axis, this implies that the leading nonanalytic term in $\Sigma(i\omega_n)$ is $\sim |\omega_n|^5$. Now consider the self-consistency (16). The analytic terms in Σ lead to apparently innocuous frequency and mass renormalizations, while the nonanalytic terms vanish so rapidly that they do not modify the assumed low-frequency form $Q^{aa}(i\omega_n) \sim |\omega_n|$; our initial assumption is therefore self-consistent. Terms higher order in $1/M$ have even weaker nonanalytic contributions. Thus there are no modifications to the leading critical properties, order by order to all orders in $1/M$. At finite temperatures, the gap in the spectrum at $g = g_c$ in (14) and (15) will of course be filled in by thermal excitations; however, all such contributions appear in the form of subdominant corrections to scaling, i.e., they are suppressed by additional powers of $(\omega$ or $T)/(J$ or $g)$. Similar considerations also apply to the threshold singularity in the $T = 0$ spectrum of the spin-fluid phase, where again the $M = \infty$ form survives.

This behavior can be better understood using a classical statistical mechanics point of view, in which the system is viewed as a classical one-dimensional spin system with a long range interaction $Q^{aa}(\tau)$; having solved the model we can then require the self-consistency (4). Our results above imply that the critical point of the quantum phase transition corresponds to a spin system with $Q(\tau) \sim 1/\tau^2$ for large τ ($1/\tau^2$ is the Fourier transform of $|\omega|$). We may consider a lattice discretization of τ, and also replace the fixed length spins by M-component soft spins $\vec{S}$ with a Landau-Ginzburg potential local in time. Thus we are led to a model with action whose continuum limit is

$$S = -\int d\tau\, d\tau'\, Q(\tau - \tau')\vec{S}(\tau) \cdot \vec{S}(\tau')$$
$$+ \frac{1}{2}\int d\tau \left[\frac{1}{g}[\partial_\tau \vec{S}(\tau)]^2 + r\vec{S}^2(\tau) + u(\vec{S}^2)^2\right]. \qquad (17)$$

where g, r, u are constants. This classical spin system, with $Q(\tau) \sim 1/\tau^{1+\sigma}$, was studied many years ago [17]. These authors found a high-temperature paramagnetic phase with power-law spin correlations, and a transition to a low-temperature ordered state if $\sigma < 1$, or if $M = 1$, $\sigma = 1$. In the high-temperature phase they found $\langle \vec{S}(\tau) \cdot \vec{S}(\tau') \rangle \sim 1/\tau^{1+\sigma}$ which is also the result obtained from the leading term in the high-temperature expansion (expansion in powers of Q). Throughout the high-temperature phase the spin-spin correlation exponent is unmodified by higher-order terms. As $Q(\tau)$ and $\langle \vec{S}(\tau) \cdot \vec{S}(\tau') \rangle$ have the same asymptotic decay, it is evident that at least self-consistency in the value of σ can be achieved for any σ. The result that $\sigma = 1$ corresponds to the quantum phase transition can be traced to the ω_n^2 in (6) or (17) which is generically present as the leading analytic ω_n dependence. It thus has nothing to do with the critical point of the one-dimensional system; the quantum-critical point corresponds to a point in the high-temperature phase of the classical spin model. With the choice $\sigma = 1$, the other critical properties then follow; the logarithmic violation of scaling comes in this model from summing tadpole diagrams in the S^4 interaction. These arguments are valid for all M including $M = 1$ (the transverse Ising case).

This paper has presented a soluble model with infinite-range interactions which displays a quantum phase transition from a spin-glass to a spin-fluid phase. Further theoretical work on the extension of these results to finite-range interactions is clearly required. The current experimental measurements on transverse-field Ising magnet $\mathrm{LiHo}_x\mathrm{Y}_{1-x}\mathrm{F}_4$ [7] have not yet examined the quantum-critical region ($\omega \sim T$); such measurements should offer a useful test of our predictions for the scaling properties of $\chi''(\omega)$.

Note added.—J. Ye has examined H in finite dimensions (d) and finds the preliminary results $\nu = 1/4$, $z = 2$ for $d > d_u = 6$.

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