Interaction of Surface Waves with Vorticity in Shallow Water

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(Received 8 February 1993)

Vortical flows in shallow water interact with long surface waves by virtue of the nonlinear terms of the fluid equations. Analytical formulas are derived that quantify the spontaneous generation of such waves by unsteady vorticity as well as the scattering of surface waves by vorticity. In a first Born approximation the radiated surface elevation is linearly related to the Fourier transform of the vorticity. The "dislocated" wave fronts that are analogous to the Aharonov-Bohm effect are obtained as a special case.

PACS numbers: 47.35.+i, 47.32.−y

Nonlinear effects in one-dimensional surface wave propagation over a shallow fluid have been widely studied in the past two decades in association with solitons [1]. What happens in two dimensions is, by comparison, far less understood [2]. The effect of vorticity [3] and of submerged bodies [4] on free surface motion has also been studied, again in one dimension, and the vortex-surface interaction is the subject of much current research [5].

In a different vein, in recent years it has become apparent that the nonlinear interaction between sound and vorticity at low Mach numbers can be profitably understood in terms of concepts borrowed from classical field theory of sources and waves [6]. This has allowed for an understanding of vortex dynamics in a slightly compressible fluid [7] and has suggested the use of ultrasound as a probe of vorticity in both ordered and disordered flows [8]. Recent experimental results have confirmed the soundness of this proposal [9]. On the other hand, it is well known [10] that the nonlinear propagation of sound obeys similar equations as the nonlinear propagation of surface waves in shallow water, and it is the purpose of this paper to apply the concepts that have been of use to understand the interaction of sound with vorticity to the understanding of the interaction of two-dimensional surface waves with ordered or disordered vorticity in shallow water.

Consider an incompressible fluid of undisturbed uniform depth h moving with velocity $\mathbf{v}$ in a uniform gravitational field g. We shall refer to it as shallow water although of course it need not be water. The free surface will be described by

$$z=\bar{\zeta}(x,y,t)\,,$$

where $\bar{\zeta}=h+\zeta(x,y,t)$, and ζ is the deviation of the surface away from the horizontal, whose typical length scales will be supposed to be much longer than h, thus allowing for the neglect of surface tension. In shallow water vertical variations of $\mathbf{v}_\perp$ are neglected and the governing equations are

$$\begin{aligned}&\frac{\partial\bar{\zeta}}{\partial t}+\nabla_\perp\cdot(\bar{\zeta}\mathbf{v}_\perp)=0\,,\\&\frac{\partial\mathbf{v}_\perp}{\partial t}+(\mathbf{v}_\perp\cdot\nabla_\perp)\mathbf{v}_\perp=-g\nabla_\perp\bar{\zeta}\,,\end{aligned}\tag{1}$$

where the subindex $\perp$ means horizontal, or x-y, components and will be omitted from here on. Equations (1) are supposed to be evaluated at $z=\bar{\zeta}$, and viscosity has been neglected. The boundary condition at the bottom is then $v_z(z=0)=0$. These equations are similar to, but not identical with, the equations for a compressible, adiabatic bulk flow and they can be combined to yield

$$\frac{\partial^2\zeta}{\partial t^2}-gh\nabla^2\zeta=h\nabla\cdot[(\mathbf{v}\cdot\nabla)\mathbf{v}]-\frac{\partial}{\partial t}[\nabla\cdot(\mathbf{v}\zeta)]\,.\tag{2}$$

This is a wave equation for the surface waves with a source term due to the nonlinear couplings. One can think then of at least two situations of interest when there is a bounded (two-dimensional) vortical flow with typical velocities very small compared to $\sqrt{gh}$: One is the spontaneous generation of surface waves by this flow, in analogy with the aeroacoustic generation of sound by vortical flows. The other is the scattering of surface waves by the flow.

The reason for the surface wave generation by a bounded flow is that, although surface deformations will not be the dominant effect at the source, far away from it they will decay, because of their wavelike nature, like $(\text{distance})^{-1/2}$, while the two-dimensional flow associated with the source, being incompressible, will decay like $(\text{distance})^{-1}$. A formal solution of Eq. (2) can be written as a convolution of the right-hand side with the Green function for the two-dimensional wave equation. If surface elevations are neglected at the source, an acceptable approximation, the source looks almost like the Lighthill source term [11]. The similarity is, however, not an identity because in the present case $\nabla_\perp\cdot\mathbf{v}_\perp\neq0$.

In the spontaneous generation of surface waves by a flow of bounded (in space) vorticity the second term in the right-hand side can safely be neglected because surface deformations are supposed to be negligible at the source and one obtains the following expression for the far field surface waves spontaneously radiated by such a flow:

$$\zeta=\frac{h}{c^2}(\boldsymbol{\omega}\wedge\mathbf{u})*\nabla G\,,$$

where $\mathbf{u}$ is a perfectly two-dimensional incompressible

0031-9007/93/70(25)/3896(4)$06.00

flow and $\boldsymbol{\omega}=\nabla\wedge\mathbf{u}$ is its (two-dimensional) vorticity. This is in very close analogy to the formulas for aeroacoustic sound derived by Powell and Howe [12] which are best manipulated in Fourier space [13], where

$$G(\mathbf{x},\nu)=\frac{i}{8\pi}H_0^+\left(\frac{\nu|\mathbf{x}|}{c}\right)$$

is a Hankel function with outgoing wave boundary condition.

The case of scattering of a plane wave can be studied with ease in the case that wave frequencies ν are high compared to any frequencies Ω associated with the target flow, and fluid velocities $\mathbf{v}_s$ associated with the surface wave are small in comparison with those associated with the vortical flow, $\mathbf{u}$, which in turn are supposed to be small compared with $c\equiv\sqrt{gh}$. To this end we write

$$\mathbf{v}=\mathbf{v}_s+\mathbf{u}\,, \tag{3}$$

where $\mathbf{u}$ is a perfectly two-dimensional incompressible vortical flow with a perfectly flat surface and $\mathbf{v}_s$ ($v_s\ll u$) is what is needed for $\mathbf{v}$ to satisfy Eqs. (1). Moreover, the differential equation (2) can be turned into the integral equation

$$\zeta=\zeta_{\rm inc}+G*s\,, \tag{4}$$

where $\zeta_{\rm inc}$ is in the incident plane surface wave and $G*s$ is a convolution of the Green function G for the wave equation with the source

$$s=h\nabla\cdot[(\mathbf{v}\cdot\nabla)\mathbf{v}]-\frac{\partial}{\partial t}[\nabla\cdot(\mathbf{v}\zeta)]\,. \tag{5}$$

Substitution of the decomposition (3) into the source (5) and neglecting the horizontal components of the vorticity, a valid approximation in shallow water, as well as terms quadratic in v_s, leads to a source that is the sum of three components:

$$s=s_1+s_2+s_3\,,$$

where

$$s_1=h\nabla\cdot(\boldsymbol{\omega}\times\mathbf{v}_s)\,,$$

$$s_2=h\left[\nabla^2(\mathbf{u}\cdot\mathbf{v}_s)-\frac{\partial}{\partial t}(\mathbf{u}\cdot\nabla)\zeta\right],$$

$$s_3=h\nabla\cdot[(\mathbf{u}\cdot\nabla)\mathbf{u}]\,.$$

The source s_3 corresponds to the spontaneous generation of surface waves discussed above and it will not be considered further in the scattering context since the waves so generated are of much lower frequency than the incoming and scattered waves. For weak surface waves the scattered amplitude will be much weaker than the incident one and Eq. (4) can be solved in a first Born approximation, that is, replacing v_s by the incident plane wave value

$$\mathbf{v}_{\rm inc}=v_0\hat{\mathbf{n}}\cos(\mathbf{k}_0\cdot\mathbf{x}-\nu_0 t)\,,$$

for which velocity at the surface and surface elevation are related by

$$\frac{\partial\mathbf{v}}{\partial t}=-g\nabla\zeta\,.$$

Using the fact that, in the far field

$$\nabla G\approx-\frac{\hat{\mathbf{x}}}{c}\frac{\partial G}{\partial t}$$

and integrating by parts it is tedious but straightforward to show that if the scattered wave is written as

$$\zeta_{\rm scatt}=\zeta_1+\zeta_2\,,$$

where $\zeta_a=G*s_a$, $a=1,2$, then

$$\zeta_1=(\hat{\mathbf{k}}_0\cdot\hat{\mathbf{x}}-1)\zeta_2+\frac{1}{gc^2}G*\left[\frac{\partial\mathbf{u}}{\partial t}\cdot\frac{\partial\mathbf{v}_{\rm inc}}{\partial t}-\hat{\mathbf{k}}_0\cdot\hat{\mathbf{x}}\frac{\partial}{\partial t}\left(\frac{\partial\mathbf{u}}{\partial t}\cdot\mathbf{v}_{\rm inc}\right)\right].$$

The second term on the right is of order $\sim\Omega\nu u v_0/c^2$ and can be neglected with respect to ζ_2, which is of order $\sim k_0^2 u v_0$, in the case under consideration of incident frequencies much higher than typical frequencies associated with the vortical target: $\Omega\ll\nu_0$.

We are thus finally led to the relation

$$\zeta_{\rm scatt}=\frac{\cos\theta}{g(\cos\theta-1)}G*[\nabla\cdot(\boldsymbol{\omega}\times\mathbf{v}_{\rm inc})]\,, \tag{6}$$

where θ is the scattering angle and $\boldsymbol{\omega}$ the vorticity. Using the far-field expression for G,

$$G(\mathbf{x},\nu)\sim\frac{1}{4\pi}\left(\frac{c}{2\pi\nu|\mathbf{x}|}\right)^{1/2}\exp i\left(\frac{\nu|\mathbf{x}|}{c}+\frac{\pi}{4}\right),$$

this expression becomes

$$\zeta_{\rm scatt}(\mathbf{x},\nu)=h\left(\frac{\sin\theta\cos\theta}{\cos\theta-1}\right)\left(\frac{\pi^3\nu}{2c^3|x|}\right)^{1/2}\left(\frac{v_0}{c}\right)e^{i(\nu|x|/c+3\pi/4)}\hat{\omega}(\mathbf{q},\nu-\nu_0)\,, \tag{7}$$

which is the promised relation between scattered surface wave amplitude at position $\mathbf{x}$ and frequency ν in terms of $\hat{\omega}$, the Fourier transform of vorticity. The incident wave has (velocity) amplitude v_0 and frequency ν_0; $\mathbf{q}$ is the momentum transfer. Note that the apparent divergence of this expression for small scattering angles is an artifact of taking an incident plane wave which is, technically, infinite. The angular dependence will break down for wavelengths λ at angles such that

$$\sin\theta \sim \lambda/L ,$$

where L is the length of the order of the spatial extent of the incident wave. Formula (7) suggests a possible nonintrusive way to probe vortical flows in shallow water using surface wave scattering.

Berry *et al.* [14] have remarked that a plane wave of wave vector $\mathbf{k}$ propagating along the x direction incident on a stationary point vortex will give rise to surface deformations that can be locally (as opposed to globally) described by a multivalued phase:

$$\zeta \sim e^{i(kr\cos\theta + \nu t + \alpha\theta)}$$

corresponding to wave fronts exhibiting a dislocation at the position of the vortex. Here (r,θ) are polar coordinates centered at the vortex and $\alpha = \nu\Gamma/c^2$, where Γ is the vortex circulation. For the dislocation to be noticeable α must be of order 1. It is of interest to see how this is related to the results reported above, which are not immediately applicable since the scattering of surface waves by vortical flows was worked out for high frequency waves, for which this last condition is violated. Indeed, at a distance $\sim\lambda$ from a point vortex the frequency of the vortical flow is $\Omega \sim \Gamma/\lambda^2$ which is $\sim \nu$ when $\alpha \sim 1$.

In the presence of a steady divergenceless background flow $\mathbf{U}$ giving rise to surface deformations ζ_0, small velocity $\mathbf{v}$ and surface deviations ζ will obey the equations, easily derived from (1),

$$\left(\frac{\partial}{\partial t} + \mathbf{U}\cdot\nabla\right)\zeta + h\nabla^2\tilde{\phi} = 0 ,$$
$$\left(\frac{\partial}{\partial t} + \mathbf{U}\cdot\nabla\right)\tilde{\phi} + g\zeta = 0 , \tag{8}$$

where $\tilde{\phi}$ is the velocity potential for the velocity, $\mathbf{v} = \nabla\tilde{\phi}$, which exists locally outside the vortex core. Quadratically small terms have been neglected and position is supposed to be sufficiently far away from the vortex core that gradients are dominated by wave vectors. Under the additional assumption that $U^2 \ll c^2$, Eqs. (8) yield

$$\frac{\partial^2\tilde{\phi}}{\partial t^2} + 2\mathbf{U}\cdot\nabla\frac{\partial\tilde{\phi}}{\partial t} - c^2\nabla^2\tilde{\phi} = 0 . \tag{9}$$

Note now that the velocity field outside a vortex is the gradient of a multivalued scalar potential which is proportional to the polar angle centered at the vortex:

$$\mathbf{U} = \nabla\left(\frac{\Gamma\theta}{2\pi}\right) .$$

Looking now for time harmonic solutions in the form

$$\tilde{\phi} = \phi e^{i\nu t} e^{i\Phi}$$

we have that with

$$\Phi = \frac{\nu\Gamma\theta}{2\pi c^2} ,$$

ϕ obeys the equation

$$\left(\nabla^2 + \frac{\nu^2}{c^2}\right)\phi = 0 ,$$

giving rise to the dislocated wave fronts of Berry *et al.* [14].

In addition to the dislocated incident wave there is a scattered wave. In the case of the Aharonov-Bohm effect [14] the latter can be calculated using vanishing boundary conditions at the "vortex" core. The scattering mechanism in the fluid case is different, its origin being not in impenetrable boundary conditions but in the nonlinear interaction terms that were neglected above. The computation of this effect offers an interesting challenge which is, however, outside the scope of the present paper.

To conclude, we have studied the nonlinear interaction between surface waves (for a two-dimensional surface) and vorticity in shallow water. Analytical formulas for the generation and scattering of such waves by any (two-dimensional) vortical flow have been derived under the following assumptions: (a) Time scales are such that viscosity can be neglected. (b) Length scales are such that surface tension can be neglected. (c) For scattering, the frequency of the incident wave is high by comparison with the (inverse of) the time scale of the vortical flow, and the particle velocity associated with the wave is supposed to be small compared with the velocity of the vortical flow, which in turn is supposed to be small compared to $\sqrt{gh}$, the phase velocity of the waves. Dislocated wave fronts analogous to the Aharonov-Bohm effect have been obtained as a special case. The question naturally arises as to what happens when the fluid is not shallow. The point of view presented in this paper, namely, that of studying the nonlinear interaction between surface motions and vorticity by way of successive approximations using ideas borrowed from classical field theory, may be of use in studying this problem.

This work was supported by DTI Grant No. E2854-9244, FONDECYT Grant No. 1265-91, and CEC Contract No. CI1*-CT91-0947.

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