Weak Localization and Integrability in Ballistic Cavities

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We demonstrate the existence of an interference contribution to the average magnetoconductance, $\overline{G}(B)$, of ballistic cavities and use it to test the semiclassical theory of quantum billiards. $\overline{G}(B)$ is qualitatively different for chaotic and regular cavities (saturation versus linear increase) which is explained semiclassically by the differing classical distribution of areas. The magnitude of $\overline{G}(B)$ is poorly explained by the semiclassical theory of coherent backscattering (elastic enhancement factor); interference between trajectories which are not exactly time reversed must be included.

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The main approach to relating the quantum properties and the classical mechanics of a system is semiclassical theory which expresses a quantum property in terms of the interference between certain classical paths [1]. Though numerical summation over paths can be used to evaluate semiclassical quantities [2], the analytical work to date includes only the interference between symmetry related paths—the diagonal approximation (DA)—while interference between paths unrelated by symmetry has remained largely intractable. In spite of this, many analytical results have been obtained; in the most studied example, the occurrence of chaotic or integrable classical dynamics determines the nature of the fluctuations in the quantum density of states [1].

In this paper we demonstrate a new quantum interference effect—a change in the *average* conductance of a ballistic cavity upon applying a magnetic field—caused by breaking time-reversal symmetry. Our semiclassical theory builds on those for studying quantum chaotic scattering [3–6] and fluctuations in the quantum conductance [7]. Within the DA, we find that the average magnetoconductance, $\overline{G}(B)$, is qualitatively different for chaotic and regular cavities: saturation versus linear behavior. Our numerical results show good agreement with this prediction, but also show that off-diagonal terms—interference between paths unrelated by symmetry—make a large contribution to the magnitude of $\overline{G}(B)$, in some cases eliminating it. While one expects off-diagonal terms to be important for very long paths [1, 8], the contribution of such trajectories in our cavities is small because particles escape. Thus, this quantum interference effect highlights a surprising failure of the diagonal approximation in semiclassical quantum theory.

The change in the average conductance upon applying a magnetic field is, of course, well known in disordered metallic conductors and is called weak localization [9]. Despite strong similarities between ballistic chaotic systems and disordered systems, we show that weak localization in ballistic systems is richer than in the diffusive regime: In addition to the different behavior of chaotic and regular cavities, the effect of spatial symmetry and short nonergodic paths is large.

Since transport coefficients of microstructures directly measure scattering probabilities, they offer the possibility of direct experimental tests of "quantum chaos" [4, 7]. In fact, Marcus *et al.* recently reported [10] an experimental study of conductance fluctuations in which a difference between nominally chaotic and regular shapes was observed. They also noted a large magnetoresistance peak at $B = 0$ and suggested a connection to weak localization.

We compute the conductance of a cavity with two leads by relating it to the transmission intensity through $G = (e^2/h)T$. Figure 1 shows $T(k)$ for a half-stadium structure calculated [11] using a square lattice and the recursive Green function method. To reduce nonuniver-

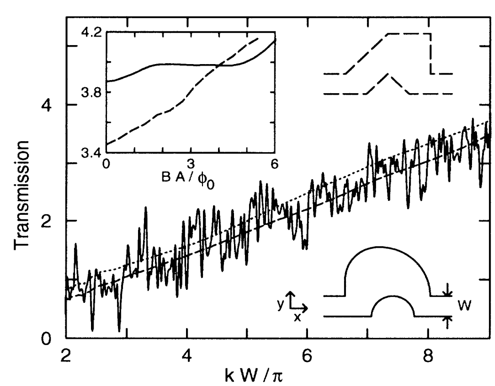

FIG. 1. Transmission coefficient as a function of wave vector for the half-stadium structure shown in the bottom right. The $T = 0$ fluctuations (solid) are eliminated by smoothing using a temperature ($T = 6$ K for $W = 0.5\ \mu$m) which corresponds to twenty correlation lengths. The offset of the resulting $B = 2\alpha_{\rm cl}\phi_0$ curve (dotted) from that for $B = 0$ (dashed) demonstrates the average magnetoconductance effect. Inset: Smoothed transmission coefficient as a function of the flux through the cavity ($kW/\pi = 9.5$) showing the difference between the chaotic (solid) and regular (dashed) structures.

sal effects, we study an asymmetric structure in which a stopper blocks the directly transmitted paths; simpler structures are discussed below. The rapid fluctuations of T were studied previously [3, 7, 12]; here we concentrate on the average conductance. A natural averaging procedure is to convolve with the derivative of the Fermi function to simulate nonzero temperature. The shift between the dashed and dotted curves in Fig. 1 is the average magnetoconductance that is the subject of this paper.

The average $T(B)$ in the inset of Fig. 1 shows that this ballistic weak-localization effect can be substantial. Traces for two structures are shown: a half-stadium structure and a similar structure with straight rather than curved sides. The classical dynamics in the half stadium is chaotic [13, 14] while that in the straight-sided structure is regular. In the straight-sided structure the dynamics cannot be ergodic because the angle of a path exiting from the cavity must be related to the angle at entry through reflection from either vertical, horizontal, or diagonal walls. The difference between the behavior of these two structures—saturation versus linear increase —shows that ballistic weak localization distinguishes between chaotic and regular classical dynamics.

As a starting point for a semiclassical theory, we write $T(k)$ in terms of classical paths which traverse the cavity [7, 15]. For leads of width W which support $N = \mathrm{Int}(kw/\pi)$ modes, the total transmitted intensity summed over incoming (m) and outgoing (n) modes is

$$T(k) = \sum_{n=1}^{N}\sum_{m=1}^{N} T_{nm} = \frac{1}{2}\frac{\pi}{kW}\sum_{n,m}\sum_{s}\sum_{u} F_{n,m}^{s,u}(k), \qquad (1)$$

$$F_{n,m}^{s,u}(k) = \sqrt{\tilde{A}_s\tilde{A}_u}\exp\left[ik(\tilde{L}_s - \tilde{L}_u) + i\pi\phi_{s,u}\right]. \qquad (2)$$

The paths in the sum, labeled s and u, are those which enter at (x, y) with fixed angle $\sin\theta = \pm m\pi/kW$ and exit at (x', y') with angle $\sin\theta' = \pm n\pi/kW$. In terms of the action S_s, the phase factor is $k\tilde{L}_s = S_s/\hbar + ky\sin\theta - ky'\sin\theta'$ plus an additional phase, $\phi_{s,u}$, associated with singular points in the classical dynamics [11]. The prefactor is $\tilde{A}_s = |\,(\partial y/\partial\theta')_\theta\,|\,/(W\cos\theta')$.

Because the classical transmission coefficient is proportional to kW/π, we expect a linear contribution to the average quantum transmission and call the slope $\mathcal{T}$. By averaging $T(k)/(kW/\pi)$ over all k [16], one can show [11] that only terms with paired paths, $s = u$, contribute to $\mathcal{T}$. The result,

$$\mathcal{T} = \frac{1}{2}\int_{-1}^{1} d(\sin\theta)\int_{-1}^{1} d(\sin\theta')\sum_{s(\theta,\theta')}\tilde{A}_s, \qquad (3)$$

is the classical probability of transmission. Thus, the leading order term in the average quantum conductance is the classical conductance [11].

The quantum corrections are best discussed in terms of the reflection coefficient, $R = N - T$, for which identical semiclassical expressions hold in terms of reflected paths. The quantum corrections to R are

$$\delta R = \frac{1}{2}\frac{\pi}{kW}\left[\sum_{n}\sum_{s\neq u} F_{n,n}^{s,u} + \sum_{n\neq m}\sum_{s\neq u} F_{n,m}^{s,u}\right], \qquad (4)$$

where we have separated the terms diagonal in mode number, $\delta R_D \equiv \sum_{n=1}^{N}\delta R_{nn}$, from the off-diagonal terms. From results in disordered conductors, one expects that coherent backscattering will influence $R_D(B)$ [9]. Previous work [1, 3–5] has shown that a typical diagonal reflection element is larger than a typical off-diagonal element by a factor of 2 when the system is time reversal invariant, a ratio known as the elastic enhancement factor. We pattern our discussion of δR_D after previous semiclassical treatments of the elastic enhancement factor [4, 5, 17].

There is a natural procedure for finding the average of δR_D over all k [16] denoted $\langle\delta R_D\rangle$: The sum of N reflection elements each with $|\sin\theta| = |\sin\theta'|$ can be converted to an integral over angle, $(\pi/kW)\sum_n \rightarrow \int d(\sin\theta)$. Then, the only k dependence is in the exponent so that the average eliminates all paths except those for which $\tilde{L}_s = \tilde{L}_u$ exactly. In the absence of symmetry $\tilde{L}_s = \tilde{L}_u$ only if $s = u$, but for time-reversal symmetry ($B = 0$) $\tilde{L}_s = \tilde{L}_u$ also if u is s time reversed. It is crucial to consider the diagonal part of R in order that the symmetry related paths satisfy the same boundary conditions on the angles. A weak magnetic field does not change the classical paths appreciably but does change the phase difference of the time-reversed paths by $(S_s - S_u)/\hbar = 2\Theta_s B/\phi_0$ where $\Theta_s \equiv 2\pi\int_s \mathbf{A}\cdot d\mathbf{l}/B$ is the effective area enclosed by the path ($\times 2\pi$) and $\phi_0 = hc/e$. Thus we obtain

$$\langle\delta R_D(B)\rangle = \frac{1}{2}\int_{-1}^{1} d(\sin\theta)\sum_{s(\theta,\theta),s(\theta,-\theta)}\tilde{A}_s e^{i2\Theta_s B/\phi_0}, \qquad (5)$$

a k-independent contribution to $\overline{G}(B)$ which depends essentially on only the semiclassical approximation. Thus the average over all k eliminates all but the symmetry related paths: We have shown that the contribution to $\langle\delta R\rangle$ which is diagonal in modes is also diagonal in terms of paths.

Evaluation of the second term in Eq. (4) is more difficult because this term necessarily involves paths which are not exactly related by symmetry: The contribution which is off diagonal in modes is off diagonal in paths. These terms do not have a simple limiting procedure as for $\langle\delta R_D\rangle$ above, and it is currently not known how to evaluate them. However, our numerical results below show that such off-diagonal terms are important.

For a chaotic system, one can estimate both the magnitude and field scale of $\langle\delta R_D\rangle$. First, if the mixing time for particles within the cavity is much shorter than the escape time, no preference is shown to scattering through any particular angle. To be precise, we assume the outgoing $\sin\theta'$ are distributed uniformly for an arbitrary distribution of ingoing trajectories. Classical simulations show that this is approximately obeyed for the structure in Fig. 1 and improves if the opening to the leads is made smaller. Thus, we replace the sum over backscattered paths in Eq. (5) by an average over all $\sin\theta'$ and find that the resulting expression for $\langle \delta R_D(B=0)\rangle$ is the same as that for $\mathcal{R}$, defined as for $\mathcal{T}$ in Eq. (3). Second, to estimate the field scale, we group the backscattered paths by their effective area and average over the distribution of this area, $N(\Theta,\theta)$,

$$\langle \delta R_D(B)\rangle \approx \int_{-\pi/2}^{\pi/2} d\theta \int_{-\infty}^{\infty} d\Theta\, N(\Theta,\theta) e^{i2\Theta B/\phi_0}. \qquad (6)$$

Previous theoretical and numerical work has shown that in a chaotic system $N(\Theta) \propto \exp(-\alpha_{\rm cl}|\Theta|)$ for large Θ independent of θ, where $\alpha_{\rm cl}$ is the inverse of the typical area enclosed by a classical path [4, 7, 14, 18]. Using this form for all Θ (an accurate approximation for structures similar to those studied here [7, 12]) yields a Lorentzian dependence on B. Combining this result with that for the magnitude above, we find

$$\langle \delta R_D(B)\rangle = \mathcal{R}/[1+(2B/\alpha_{\rm cl}\phi_0)^2]. \qquad (7)$$

We emphasize that this is obtained from the semiclassically exact Eq. (5) using two controllable approximations, uniformity and the exponential area distribution. Note that the field scale can be much smaller than ϕ_0 through the area of the cavity.

For a regular cavity, we estimate $\langle \delta R_D(B)\rangle$ by using the appropriate $N(\Theta,\theta)$, following work in the energy-time domain [6, 19]. For a fixed θ, we suppose that the trajectories are ergodic in real space and therefore that $N(\Theta,\theta)$ is exponential as in the chaotic case. However, unlike the chaotic case, the rate of decay depends strongly on θ and may vanish. This rate of decay is proportional to the square root of the typical escape rate γ [4, 14, 18], $N(\Theta,\theta) \propto \exp[-c|\Theta|\sqrt{\gamma(\theta)}]$, so that the points where $\gamma \approx 0$ dominate the large Θ behavior. For the regular structure of Fig. 1, γ vanishes linearly at $\theta = \pm\pi/2$ as particles are injected close to a periodic orbit, and $\int d\theta\, N(\Theta,\theta) \propto 1/\Theta^2$. Taking the Fourier transform, we conclude that $\langle \delta R_D(B)\rangle \propto |B|$ for small $|B|$. (The unphysical cusp at $B=0$ is caused by deviations from $1/\Theta^2$ at very large Θ [6].) *Thus, a qualitative difference between chaotic and regular cavities results from the different classical distributions of effective areas.*

In Fig. 2 we compare the semiclassical predictions for the chaotic case to numerical results. We calculate all the R_{mn} and hence can extract $R_D = \sum R_{nn}$. The results for the structure with the stopper are in accord with the semiclassical theory: δR_D is approximately independent of k, its magnitude is within 30% of $\mathcal{R}$, and the elastic enhancement factor is 1.97 ± 0.07 at $B=0$ and 0.99 ± 0.02 at $B/\alpha_{\rm cl}\phi_0 = 2$. The somewhat low magnitude of δR_D may result from the difficulty in numerically achieving an escape time much larger than the mixing time. Some net

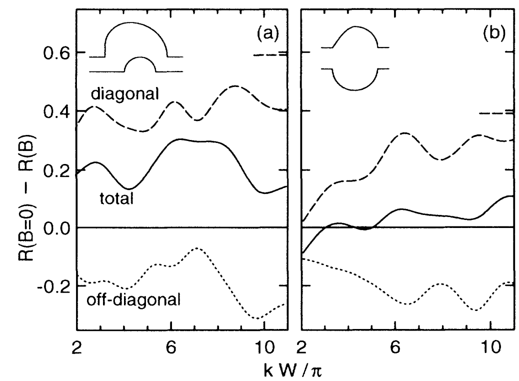

FIG. 2. Change in the total reflection coefficient (solid), as well as the diagonal (dashed) and off-diagonal (dotted) parts, upon changing B from 0 to $2\alpha_{\rm cl}\phi_0$. The curves are smoothed using a window of $1.5kW/\pi$ [thirty correlation lengths for panel (a), fifteen for (b)]. The dashed ticks on the right mark the classical value of $\mathcal{R}$. Note the roughly k-independent behavior of the curves in (a) and the large contribution of the off-diagonal reflection coefficients to the total weak-localization effect.

variation as a function of k occurs in the structure with direct transmission paths [Fig. 2(b)] as well as a smaller total magnitude, indicating that short structure-specific paths can have a large effect on $\overline{G}(B)$.

Figure 2 shows that there is a large change in the off-diagonal terms of *opposite* sign to $\langle \delta R_D\rangle$, a result not anticipated by the semiclassical theory above. *Thus weak localization is not equivalent to the coherent backscattering (elastic enhancement) effect.* This distinction does not appear to have been appreciated in much of the literature [4, 5, 11, 17], including that on disordered systems [9]. The importance of off-diagonal contributions is even more apparent for T since there are *no* time-reversal symmetric paths in its semiclassical expression. For the density of states of closed systems, Berry has shown that off-diagonal terms must be included for very long paths, paths longer than the inverse level spacing, while the DA is sufficient at shorter scales [8]. In our cavities, the path length is limited by the dwell time which is much smaller than the inverse level separation, so these off-diagonal interference effects are not directly relevant to the deviations from the DA that we see. Thus, while the DA is adequate for the density of states at these relatively short times [1, 8], it is evidently inadequate for the magnitude of transport quantities.

Despite this inaccuracy, the semiclassical theory presented here is successful in certain key respects. It shows that reflection coefficients are sensitive to B through time-reversal symmetry, relates the field scale to the average area enclosed by classical paths, and explains the difference between chaotic and regular structures in terms of the distribution of the classical area.

The semiclassical results suggest analyzing the numer-

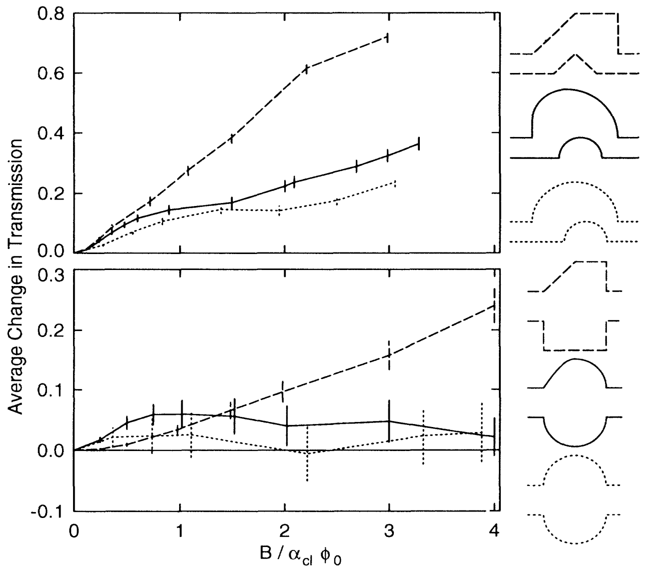

FIG. 3. Weak-localization magnitude as a function of magnetic field for the six structures shown. The magnitude is obtained from $\langle T(k,B) - T(k,B=0)\rangle_k$ with $kW/\pi \in [4,11]$. Note the difference between the chaotic and regular structures, as well as the sensitivity to symmetry in the lower panel. $\alpha_{\rm cl}$ is the inverse of the typical area enclosed by classical paths.

ical data by averaging the change in $T(k)$ as shown in Fig. 3. The top panel demonstrates the difference between chaotic and regular structures: The curves for the half stadia (chaotic) flatten out while that for the half asymmetric square (regular) increases linearly (except for very small B where it is quadratic). Not all of our chaotic structures show a clear saturation; however, all have rapidly changing magnetoconductance at small field followed by a more gradual rise. We attribute this deviation to the small size of our structures.

The lower panel of Fig. 3 shows a clear weak-localization effect for structures without stoppers. The error bars are larger than in the upper panel because of the greater variation with k produced by the direct paths. However, it is interesting, and important for experiments, that the direct paths do not mask the weak-localization effect: The difference between the chaotic and regular cavities is clear. The lower panel shows that a symmetric stadium behaves differently from the asymmetrized cases: $\overline{G}$ is nearly independent of B. The importance of spatial symmetries in determining $\overline{G}(B)$, an irrelevant effect in disordered systems, may be experimentally accessible in the ballistic regime.

Finally, we note a connection between our results and those of a random matrix theory (RMT) [20] based on taking a Hamiltonian from a Gaussian ensemble and coupling it to leads. For good coupling, the result implicit in Ref. [20] is $\langle\delta R\rangle = 0.25$ while the elastic enhancement factor yields $\langle\delta R_D\rangle = 0.5 = \mathcal{R}$. This is consistent with our findings and, because of the deviation from the DA, shows that this RMT *does* include off-diagonal contributions.

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