Spectral Density of the QCD Dirac Operator near Zero Virtuality

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We investigate the spectral properties of a random matrix model which in the large N limit embodies the essentials of the QCD partition function at low energy. The exact spectral density and its pair correlation function are derived for an arbitrary number of flavors and zero topological charge. Their microscopic limits provide the master formulae for sum rules for the inverse powers of the eigenvalues of the QCD Dirac operator, as recently discussed by Leutwyler and Smilga.

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The issue of chiral symmetry is fundamental in QCD. Lattice simulations indicate that the symmetry is spontaneously broken in the vacuum. This fact has inspired a large body of work in an attempt to describe the underlying mechanism. However, an explanation from first principles is still elusive. In a way, the spontaneous breakdown of chiral symmetry reflects directly on the way the quark states are delocalized in the vacuum [1,2], a situation reminiscent of the delocalization of electrons in solids and the onset of conductivity [3].

A decade ago, Banks and Casher [4] noted that the spontaneous breaking of chiral symmetry is related to an accumulation in the quark spectral density at zero virtuality. In other words, as pointed out by Leutwyler and Smilga [5], the spacing of the eigenvalues λ is not proportional to $1/V_4^{1/4}$ as in free space, but to $1/V_4$ (V_4 is the spacetime volume). This implies the existence of a microscopic limit of the spectral density in which the thermodynamical limit is taken for fixed values of λV_4 [6].

In the extreme long wavelength limit, the QCD partition function can be written as an integral over spacetime independent Goldstone modes [5]. This implies strong correlations between the eigenvalues of the Dirac operator in the form of sum rules. These sum rules only involve the microscopic limit of the spectral density.

We conjecture that the microscopic correlations between the eigenvalues of the Dirac operator near zero virtuality are universal and encoded in the microscopic spectral density and its fluctuations. Our conjecture is motivated by the following observations: First, the sum rules hold for the massive Schwinger model [7]. Second, they are obeyed by the Dirac operator in a liquid of instantons [6] with correlations induced by the fermion determinant [2]. Third, a random matrix model [6] can be constructed that in the large N limit reduces to the low energy limit of the QCD partition function for a given value of the vacuum angle, and thus satisfies all microscopic sum rules. Fourth, in chaotic systems microscopic fluctuations are universal and can be mapped on the invariant random matrix ensembles [8–10]. Fifth, in the theory of mesoscopic systems, the Hofstadter model has a spectral density that coincides with that of the above random matrix model in the quenched approximation [11].

The above observations suggest that a random matrix model with the general symmetries of the QCD partition might be key in understanding the general implications of chiral symmetry breaking in the QCD vacuum. The aim of the present paper is to construct the microscopic limit of its spectral density and correlations thereof. They constitute the master equations for the Leutwyler-Smilga spectral sum rules. We restrict ourselves to the simplest possible case of zero total topological charge. The model is outlined in the next section. Using the orthogonal polynomial approach in random matrix theory, we derive an explicit expression for the spectral density in the chiral limit for an arbitrary number of flavors and zero topological charge.

Consider a system of $n=N/2$ zero modes and n antizero modes with interaction given by the $n\times n$ overlap matrix T. For N_f flavors, the partition function that reflects the chiral structure of QCD is given by [6,12,13]

$$Z=\int \mathcal{D}T\, P(T)\prod_j^{N_f} \det\begin{pmatrix} m_f & iT \\ iT^\dagger & m_f \end{pmatrix}, \tag{1}$$

where the integral is over the real and imaginary parts of the matrix elements of the arbitrary complex matrix T, i.e., $\mathcal{D}T$ is the Haar measure. In agreement with the maximum entropy principle [14] the distribution function of the overlap matrix elements $P(T)$ is chosen Gaussian:

$$P(T)=\exp[(-N/2\sigma^2)TT^\dagger]\,. \tag{2}$$

The symplectic structure is a manifest consequence of chiral symmetry, and implies that the quark eigenvalues occur in pairs. The density of the zero modes, N/V_4, is taken equal to 1, which allows us to identify the spacetime volume and the number of zero modes.

The order parameter in the study of the spontaneous breaking of chiral symmetry is the quark condensate defined through

$$\Sigma_f=\langle \bar{q}_f q_f\rangle=\lim_{m_f\to 0}\lim_{N\to\infty} -\frac{1}{N}\frac{d}{dm_f}\ln Z\,, \tag{3}$$

where the chiral limit is to be taken *after* the thermodynamical limit. By writing the determinant as the product $\Pi(\lambda_n^2+m_f^2)$ one obtains the Banks-Casher formula [4],

$$\Sigma = \pi \langle \rho_c(0) \rangle_Z , \tag{4}$$

where the average $\langle \cdots \rangle_Z$ is with respect to the partition function (1). The continuum spectral density is defined by

$$\rho_c(\lambda) = \lim_{\text{all } m_f \downarrow 0} \lim_{N \to \infty} (1/N) \rho(\lambda) , \tag{5}$$

and the spectral density $\rho(\lambda)$ is

$$\rho(\lambda) = \sum \delta(\lambda - \lambda_n) , \tag{6}$$

where the eigenvalues $\pm \lambda_n$ are the nonzero eigenvalues of the overlap matrix in the chiral limit. As has been shown in [6] the parameter σ can be identified as $\sigma = 1/\Sigma$.

The sum rules recently discussed by Leutwyler and Smilga [5] for the QCD Dirac operator using chiral perturbation theory involve the average of the sums

$$\sum_n \frac{1}{N^{2p} \lambda_n^{2p}} , \quad \sum_{nm} \frac{1}{N^{p+q} \lambda_n^p \lambda_m^q} , \tag{7}$$

which can be rewritten as

$$\int_0^\infty dx \frac{\rho_s(x)}{x^{2p}} , \quad \int_0^\infty dx\, dy \frac{\rho_s(x) \rho_s(y)}{x^p y^q} , \tag{8}$$

respectively, where we have introduced the microscopic spectral density defined by

$$\rho_s(x) = \lim_{N \to \infty} (1/N) \rho(x/N) . \tag{9}$$

In this Letter our primary interest is to derive analytical expressions for the average of $\rho_s(x)$ and its pair correlation function that summarize the sum rules.

The partition function (1) can be evaluated by rewriting the matrix integration in polar coordinates. For an arbitrary complex matrix we may write [15]

$$T = U \Lambda V^{-1} , \tag{10}$$

where U and V are unitary matrices and Λ is a positive definite diagonal matrix. Since the right-hand side has N more degrees of freedom than the left-hand side, one has to impose constraints on the new integration variables. This can be achieved [15] by restricting U to the coset $\mathrm{U}(N)/\mathrm{U}(1)^N$, where $\mathrm{U}(1)^N$ is the diagonal subgroup of $\mathrm{U}(N)$. The Jacobian of this transformation, which depends only on the eigenvalues λ_k of Λ, is given by

$$J(\Lambda) = \prod_{k<l} (\lambda_k^2 - \lambda_l^2)^2 \prod_k \lambda_k . \tag{11}$$

The integrations over the eigenvalues and the unitary matrices decouple. The latter only result in an overall irrelevant constant factor and can be ignored. The eigenvalue distribution is thus given by

$$\rho_n(\lambda_1, \ldots, \lambda_n) = J(\Lambda) \prod_f \prod_k (\lambda_k^2 + m_f^2) \exp\left[-\frac{n}{\sigma^2} \sum_{k=1}^n \lambda_k^2 \right] . \tag{12}$$

The spectral density $\rho_1(\lambda)$ is obtained by integration over the remaining $n-1$ eigenvalues:

$$\rho_1(\lambda_1) = \int \prod_{k=2}^n d\lambda_k \, \rho_n(\lambda_1, \ldots, \lambda_n) . \tag{13}$$

These integrals can be evaluated with the help of the methods developed by Dyson, Mehta, and Wigner (see [9] for references). The main ingredient is to write the product over the differences of the eigenvalues as a Vandermonde determinant, i.e.,

$$\prod_{k<l} (\lambda_k^2 - \lambda_l^2)^2 = \begin{vmatrix} 1 & \cdots & 1 \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \lambda_1^{2(n-1)} & \cdots & \lambda_n^{2(n-1)} \end{vmatrix}^2 , \tag{14}$$

which up to a constant can be rewritten in terms of orthogonal polynomials P_k as

$$\begin{vmatrix} P_0(\lambda_1^2) & \cdots & P_0(\lambda_n^2) \\ \cdot & & \cdot \\ \cdot & & \cdot \\ P_{n-1}(\lambda_1^2) & \cdots & P_{n-1}(\lambda_n^2) \end{vmatrix} . \tag{15}$$

The P_k will be chosen orthogonal according to the weight function

$$\int_0^\infty d(\lambda^2) (\lambda^2 + m^2)^{N_f} \exp[(-n/\sigma^2)\lambda^2] P_k(\lambda^2 P_l(\lambda^2) = \delta_{kl} . \tag{16}$$

For $m = 0$ these polynomials are well known,

$$P_k(s) = \left[\frac{n}{\sigma^2} \frac{k!}{\Gamma(N_f + k + 1)} \right]^{1/2} L_k^{N_f} \left[\frac{sn}{\sigma^2} \right] , \tag{17}$$

where the $L_k^{N_f}$ are the associated Laguerre (Sonine) polynomials.

The determinants can be expanded according to Cramer's rule. All integrals can be performed immediately by orthogonality and, up to an overall constant, we are left with

$$\rho_1(\lambda) = \sum_{k=0}^{n-1} \frac{k!}{\Gamma(N_f + k + 1)} L_k^{N_f}(z) L_k^{N_f}(z) z^{N_f + 1/2} \exp(-z) , \tag{18}$$

where z is defined by

$$z = n\lambda^2/\sigma^2 . \tag{19}$$

The sum can be evaluated exactly with the Christoffel-Darboux formula, resulting in

$$\rho_1(\lambda) = \frac{n}{\sigma^2} \frac{n!}{\Gamma(N_f + n)} [L_{n-1}^{N_f}(z) L_{n-1}^{N_f - 1}(z) - L_n^{N_f}(z) L_{n-2}^{N_f + 1}(z)] z^{N_f + 1/2} \exp(-z) , \tag{20}$$

which, up to a normalization constant, constitutes the exact spectral density of the model (1). The microscopic limit is obtained by taking $N \to \infty$ while keeping $N\lambda = x$ fixed (remember that $n = N/2$). This can be achieved from the asymptotic relation

$$\lim_{n\to\infty} (1/n^a) L_n^a(x/n) = x^{-a/2} J_a(2\sqrt{x}) , \tag{21}$$

where J_a is the ordinary Bessel function of degree a. The result for the microscopic spectral density is

$$\rho_s(x) = (\Sigma^2 x/2)[J_{N_f}^2(\Sigma x) - J_{N_f+1}(\Sigma x) J_{N_f-1}(\Sigma x)] . \tag{22}$$

The normalization constant follows from the asymptotic behavior of the Bessel function and the Banks-Casher relation.

This formula reproduces all diagonal sum rules of Leutwyler and Smilga; e.g., the sum

$$\sum_n \frac{1}{N^{2p}\lambda_n^{2p}} \tag{23}$$

can be converted into an integral over the microscopic variable $x = \lambda N$ resulting in

$$\int_0^\infty \frac{\rho_s(x)dx}{x^{2p}} = \left(\frac{\Sigma}{2}\right)^{2p} \frac{\Gamma(2p-1)\Gamma(N_f-p+1)}{\Gamma(p)\Gamma(p+1)\Gamma(N_f+p)} . \tag{24}$$

The above spectral density thus summarizes all sum rules, e.g., sum rules for noninteger values of p. The spectral density for 0, 1, and 2 flavors are shown by the full, dashed, and dotted curves in Fig. 1.

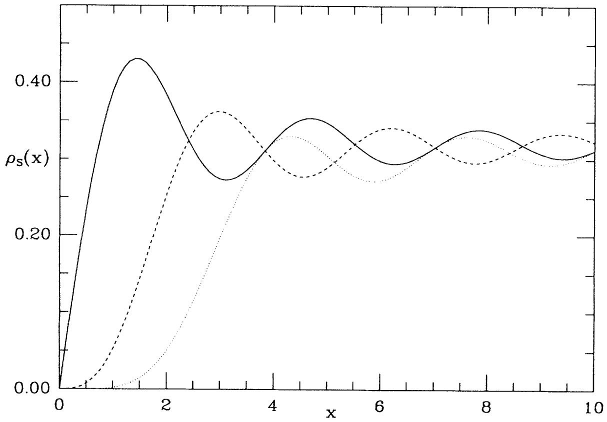

FIG. 1. The microscopic spectral density $\rho_s(x)$ as a function of the microscopic variable x for 0 (full line), 1 (dashed line), and 2 (dotted line) flavors.

The nondiagonal sum rules require correlations in the spectral density. These correlators can be calculated using similar techniques. Also, the above arguments can be extended to finite quark masses [16].

The two point correlation function $\rho_2(\lambda,\lambda')$ follows from (12) by integration over $n-2$ eigenvalues and subtracting the disconnected part. Using again the properties of the orthogonal polynomial and the Christoffel-Darboux formulae we have

$$\rho_2(\lambda,\lambda') = \left(\frac{n}{\sigma^2}\frac{n!}{\Gamma(n+N_F)}\right)^2 (zz')^{N_F+1/2} e^{-(z+z')} \times \left(\frac{L_n^{N_F}(z)L_{n-1}^{N_F}(z') - L_n^{N_F}(z')L_{n-1}^{N_F}(z)}{z-z'}\right)^2 , \tag{25}$$

with $z = n\lambda^2/\sigma^2$ and $z' = n\lambda'^2/\sigma^2$. The microscopic limit $\rho(x,y) = \lim_{N\to\infty}\rho_2(x/N, y/N)$ is obtained by using arguments similar to the ones used for the spectral density. The result is

$$\rho(x,y) = \Sigma^2 xy \left(\frac{xJ_{N_F}(\Sigma x)J_{N_F-1}(\Sigma y) - yJ_{N_F}(\Sigma y)J_{N_F-1}(\Sigma x)}{x^2-y^2}\right)^2 . \tag{26}$$

The microscopic limit (26) can be used to derive off-diagonal sum rules for the QCD Dirac operator. Indeed,

$$\left\langle \left(\sum_n \frac{1}{N^2\lambda_n^2}\right)^2 \right\rangle - \left\langle \sum \frac{1}{N^4\lambda_n^4} \right\rangle - \left\langle \sum_n \frac{1}{N^2\lambda_n^2} \right\rangle^2 = \int dx\, dy \frac{\rho(x,y)}{x^2y^2} . \tag{27}$$

We were not able to evaluate this integral analytically, but numerically the result is given by

$$\frac{\Sigma^4}{16N_f^2(N_f+1)} , \tag{28}$$

for $N_f = 1, 2, \ldots, 20$ with an accuracy of better than 1 part in 10^5 and agrees completely with Leutwyler and Smilga [5].

We have shown that the microscopic spectral density following from the Gaussian chiral ensemble of complex matrices reproduces all microscopic sum rules for the QCD Dirac operator. The corresponding random matrix model reflects solely on chiral symmetry. We have strong indications that the microscopic spectral density is universal and should be a solid property of the QCD Dirac operator near zero virtuality. The overall spectral density is not. It would be interesting to see how the former compares to the true QCD spectral density following from lattice simulations. Our approach is relevant for discussing similar issues related to QCD in two and three dimensions, as well as the spectral properties of strongly coupled QED. Finally, we note that the spectral density for zero flavors describes the spectral correlations in the Hofstadter model as used in the framework of universal conductance fluctuations [11,17,18]. The physical analogy is striking, emphasizing once more the universality of the bulk structure of chiral symmetry breaking in the QCD vacuum.

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