Fermion Doubling and Gauge Invariance on Random Lattices

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Random-lattice fermions have been shown to be free of the doubling problem if there are no interactions or interactions of a nongauge nature. On the other hand, gauge interactions impose stringent constraints as expressed by the Ward-Takahashi identities which could revive the free-field suppressed doubler modes in loop diagrams. Comparing random-lattice, naive, and Wilson fermions in two-dimensional Abelian background gauge theory, we show that indeed the doublers are revived for random lattices in the continuum limit. Some implications of the persistent doubling phenomenon on random lattices are also discussed.

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The doubling problem of lattice fermions is inevitable according to the Nielsen-Ninomiya no-go theorem [1] if the free-field action satisfies the conditions of reflection positivity, locality, global axial symmetry, and translational invariance at a fixed scale. An obvious resolution of the doubling problem is thus to relax one of those conditions to obtain, in the order listed above, non-Hermitian [2], nonlocal [3], Wilson [4], or random-lattice [5–8] fermion formulations. These formulations are all free of doublers when there are no interactions or when the interactions are of a nongauge nature [9,10]: the extra poles in the propagators are removed as the lattice spacing a decreases, leaving a single fermion mode in the continuum limit.

Gauge interactions behave very differently on account of a unique and special property. Local gauge invariance imposes severe constraints on the theory, expressed mathematically in the Ward-Takahashi identities. In particular, the fermion-gauge vertex is related to the free inverse propagator,

$$\Lambda_\mu(p,p) \sim \frac{\partial}{\partial p^\mu} G_0(p), \tag{1}$$

giving the interaction vertices mode dependency. This different coupling strength of doublers to gauge fields has been shown to revive these modes in loop diagrams, even though they are suppressed at the free-field level, in studies of some nonlocal [11] and non-Hermitian formulations [12,13]. For this reason, we investigate the issue of fermion doubling on random lattices with gauge interactions [14].

In the random-lattice approach, suitable quantities are measured on a random lattice and then averaged, either quenched or annealed, over an ensemble of lattices. Apart from the extra work involved in generating an ensemble of random lattices, this approach better approximates the scale-free rotational and translational symmetry of the continuum than regular lattices. Thus, the continuum limit may be more easily reached on random lattices than on regular lattices of the same size. More relevant to this discussion, since there is no fixed Brillouin zone, there need be no extra poles of the propagator. Even if extra poles do exist, the one-to-one correspondence between propagator poles in momentum space and zero modes is not necessarily valid since plane waves are no longer eigenstates of the Dirac operator. Alternatively, one could appeal to the fact that there is no transfer matrix on a random lattice (at least for a finite lattice), since there are no identical time slices, to argue that there may not be a clear relation between poles of the inverse propagator and the particle spectrum [8].

This expectation of no doubling on random lattices has been realized in various studies of free-field theory in both two and four dimensions [6,7]. It has also been shown that random-lattice theories with four-point interactions are doubler free [9], and the same is claimed for random-lattice theories with gauge interactions [7]. We address this claim in this Letter, emphasizing the role of gauge invariance in the suppression of doublers. We find that the random lattice does not remove fermion doubling if gauge invariance is maintained on the lattice. However, a breaking of the gauge symmetry at the lattice scale will suppress the doubler modes.

We wish to compute the correction to the fermion determinant when Abelian background gauge fields are present on a two-dimensional Euclidean random lattice,

$$-\ln \mathrm{Det}\,(G_A G_0^{-1}) = \Gamma_2 + O(g^4), \tag{2a}$$

$$\Gamma_2 = \mathrm{Tr}[(G_A^{-1}G_0 - 1) - \tfrac{1}{2}(G_A^{-1}G_0 - 1)^2], \tag{2b}$$

$$\Gamma_2 \xrightarrow{a\to 0} \int d^2x\, d^2y\, A_\mu(x)\Pi_{\mu\nu}(x,y)A_\nu(y) + O(g^4), \tag{2c}$$

where G_A^{-1} is the fermion propagator in the background gauge field A_μ, g is the gauge coupling, and $\Pi_{\mu\nu}(x,y)$ is the vacuum polarization tensor.

Γ_2 as defined by Eq. (2b) implicitly contains the fermion-gauge vertex as it appears in the lattice action, even though the explicit form is complicated and not known. Using background fields ensures our results will not be marred by internal gauge interactions; hence we

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expect to see a clean signal which increases with the number of fermion species contributing to the single fermion loop. Comparing with identical calculations for naive and Wilson fermions on square lattices, which are known to be fourfold doubling and double free, respectively, clarifies the continuum limit behavior of our random lattices.

Our lattice is constructed from a triangulated array of N fixed square lattice vertices by a sequence of Alexander "flip" moves. Each move randomly selects a quadrilateral $ABCD$, with a unique internal link AC; the internal link is deleted, and a new link BD is introduced. A flip is performed only if the local orientability of the lattice is preserved and there are no crossed links. We chose to randomize the lattice with $6N$ successful flips; see Fig. 1. The resulting lattice has a fixed size independent of flipping procedure, so measured quantities do not need to be scaled by the average link length s. However, s is a lattice dependent internal scale, which increases with flipping, making highly flipped lattices less localized and thus more sensitive to finite size effects.

The (Euclidean) fermion action is chosen such that it reduces to the naive result on lattices of regular arrangements of links. Representing the vertices and links by 2-vectors,

$$S = \sum_x \left\{ \left(\sum_l \overline{\Psi}'_x \, \gamma_\mu \, \Delta_\mu^{x,+l} \, \Psi'_{x+l} \right) - \left(\sum_l \overline{\Psi}'_{x+l} \, \gamma_\mu \, \Delta_\mu^{x,-l} \, \Psi'_x \right) + ma \, \overline{\Psi}'_x \, \Psi'_x \right\}. \quad (3)$$

At a vertex x with coordination number C_x, the differential link sum is constructed by averaging the effective square lattice derivative approximated by C_x pairs of orientation-consecutive links, $(k,l)_{i=1\ldots C_x}$,

$$\sum_l \overline{\Psi}'_x \, \gamma_\mu \, \Delta_\mu^{x,+l} \, \Psi'_{x+l} = \overline{\Psi}'_x \, C_x^{-1} \sum_{\{(k,l)_i\}} (k \otimes l)^{-1} \, \gamma \otimes [l \, \Psi'_{x+k} - k \, \Psi'_{x+l} + (k-l) \Psi'_x], \quad (4)$$

where we have used the two-dimensional antisymmetric product $k \otimes l = k_\mu l_\nu \epsilon_{\mu\nu}$, and $\Psi'_x = \sqrt{\omega_x} \, \Psi_x$ is the "area" weighted dimensionless field. The area ω_x is determined by taking 1/3 of the area of all triangles which have x as a vertex. Gauge interactions are introduced in the usual gauge-invariant manner using the link variables $U_{x,x+l} = \exp[ig \int_l A(x) \cdot dx]$. An alternative formulation $U_{x,x+l} = \exp[ig \, l \cdot A(x + l/2)]$, which is not gauge covariant under the usual continuous gauge transformations, is also considered. The resulting action is Hermitian in the Euclidean sense, local, and apart from the mass term, axially symmetric.

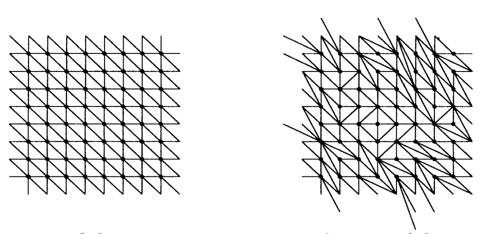

FIG. 1. A typical 8×8 lattice.

Measurements are made in a background gauge field

$$gA_\mu = \delta_{\mu,1} \frac{E\sqrt{V}}{2\pi} \cos\left(\frac{2\pi x_0}{\sqrt{V}} \right), \quad (5)$$

with fixed physical quantities: mass $= m = 0.1$, area $= V = 64$, and electric field $= E = 0.05$, for lattice spacing $= a = \{1.0, 0.5, 0.3333, 0.25\}$.

We first compute a quantity derived from the free propagator

$$f(\xi) = \mathrm{Tr}_\gamma \frac{1}{V} \int_{x,x'} (1 + \gamma_0) \, G_0^{-1}(x, x') \delta^1(x_0 - x'_0 + \xi), \quad (6)$$

evaluating the average zero-momentum real particle propagator projected along the x_0 direction. Figure 2 summarizes the calculation, clearly identifying the doubler suppression of free fermions on a random lattice, in agreement with [7]. Indeed, apart from some minor small distance fluctuations of the order of the internal scale, the random lattice result matches the continuum completely; the normalization is reproduced exactly, and masses do not need to be tuned.

The calculation of Γ_2 is complicated by its sensitivity to the structure of the lattice. This sensitivity comes

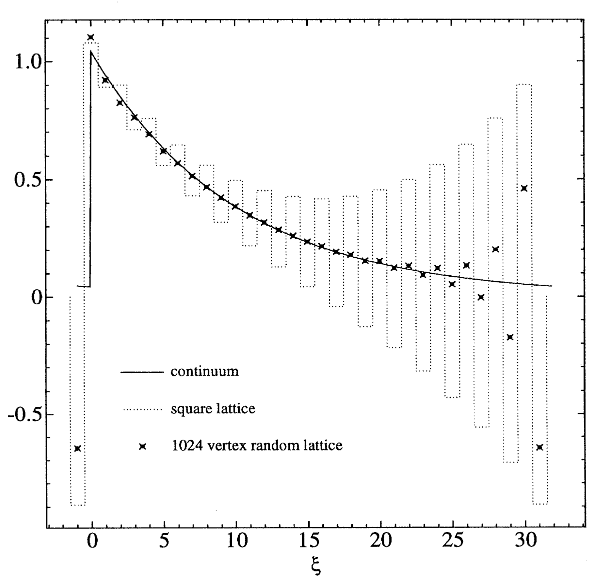

FIG. 2. Fermion propagation, $f(\xi)$.

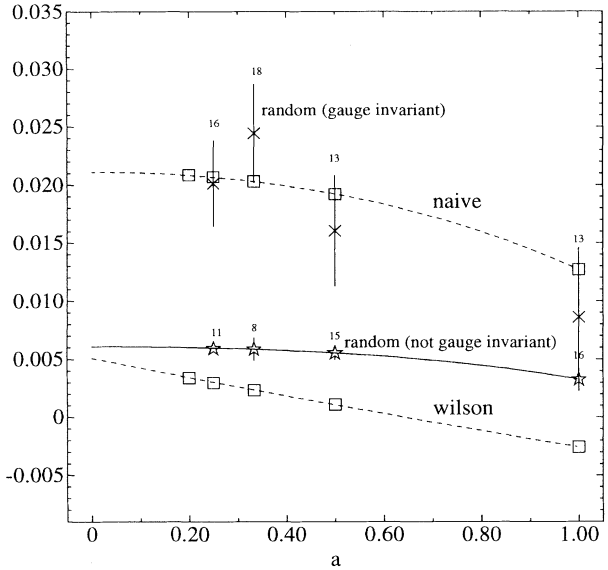

FIG. 3. Variation of Γ_2 with a.

from two sources: variations in the average link length, s, and the variation due to inequivalent arrangements of links. Hence we consider an ensemble of lattices randomized by $6N$, $12N$, $24N$, and $48N$ flips. Averaging over different lattices at fixed s, $\langle\Gamma_2\rangle_T$ shows linear variation with s; we extrapolate to $s = a$. These are the results shown in Fig. 3. The naive case approaches the continuum limit quadratically with a and the Wilson approaches 1/4 of the same result linearly, as expected. The same graph also indicates the random lattice results; the number of lattice configurations used in the extrapolation is displayed next to each point. The lattice gauge-invariant calculation is clearly more like the naive fermion than the Wilson. Moreover, as $\partial\langle\Gamma_2\rangle_T/\partial s > 0$, this serves as a lower bound. With gauge invariance broken, the converse is clearly seen; the result is certainly more like Wilson than naive. In this case $\partial\langle\Gamma_2\rangle_T/\partial s < 0$, and thus the result is an upper bound. In either case, the random-lattice results approach the continuum results more rapidly than either Wilson or naive formulations, as expected in a random-lattice approach.

It is clear from our results that there are doublers on random lattices when gauge invariance is maintained at finite lattice spacing, since the extrapolated determinant is comparable to that of naive fermions. It can also be seen that the doubling can be avoided if one gives up gauge invariance (but needs and hopes to recover it in the continuum limit).

Both lattice fermion actions are invariant under the global axial transformations. When there are doublers on random lattices, the axial anomalies are canceled in the usual manner among opposite-chirality species. When there is no doubling in the gauge-noninvariant formulation, the conserved lattice current being the Noether current of axial symmetry is, of course, not gauge invariant. Thus it cannot be identified with the continuum axial current which is invariant. It should be, instead, identified with a combination of the continuum current and a gauge-noninvariant term, whose divergence gives us the axial anomalies,

$$J^{5\mu}_{\rm lattice}(x) = J^{5\mu}_{\rm continuum}(x) - \frac{g}{2\pi}\epsilon^{\mu\nu}A_\nu(x). \tag{7}$$

We believe that the results obtained here are also applicable to other kinds of random lattices insofar as translational invariance is broken. The results of [15] of fourfold doubling on a random block lattice seem to support this claim, even though the interpretation and thus the conclusions reached there are different from ours.

Our doubling conclusion for random lattices is not only plainly disappointing but also points to some serious implications. We have extended the lattice no-go theorem and at the same time emphasized the importance of gauge invariance in the phenomenon of lattice fermion doubling. The failure of random lattices to accommodate chiral fermions could either undermine the point of view that at the Planck scale or higher the structure of spacetime is that of randomness, or, taken with other complete failures in dealing with chiral fermions, could be a hint that our understanding of chiral gauge theories is incomplete. Correspondingly, the quantization of those theories is in need of further studies. One of us has been pursuing this latter path [16].

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