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Rydberg Atoms in Curved Space-Time

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The possible use of loosely bound Rydberg atoms for remote gravimetric measurements is explored. The first-order corrections to the nonrelativistic nS and nP states for $n > 2$ are obtained for the first time. A procedure to evaluate corrections of any order is outlined and applied to the $1S$ state in a spherical symmetry. It is shown that observations of the effects described in this Letter near objects of neutron-star-like densities are possible in principle only in the absence of significant magnetic fields.

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It has been known for quite some time that the energy eigenstates of an atom are affected by the local space-time curvature [1]. These energy shifts would be detected by a locally inertial, co-moving observer close to the atom and are exclusive of further gravitational, Doppler, and cosmological shifts possibly apparent to faraway observers [2]. The typical treatment of the problem found in the literature [1] consists of using Fermi normal coordinates [3] to write the Dirac equation for a one-electron atom in curved space-time and of deriving corrective terms to the flat-space Hamiltonian. In what follows we shall be mainly concerned with the nonrelativistic limit of this approach (that is, $v/c \equiv \beta \ll 1$, not the limit of vanishing curvature). This yields a time-independent Schrödinger equation containing an effective classical geodesic deviation potential describing the tidal interaction of the atom with the gravitational field. The nonrelativistic Hamiltonian $H_{\rm NR}$, to first order in the Riemann tensor, reads

$$H_{\rm NR} = \frac{1}{2\mu}\mathbf{p}^2 - \frac{Ze^2}{r} + \frac{1}{2}\mu\, R_{0l0m}\, x^l x^m \, , \qquad (1)$$

where μ is the reduced mass, R_{0l0m} are the components of the Riemann tensor, and x^l is the position of the electron in the nucleus-centered reference frame. It should be noticed that nuclear effects, radiative corrections, and the electron-nucleus gravitational interaction are neglected, and that cgs units are used.

In the case of a hydrogenlike atom of principal quantum number n in free fall at the surface of a spherical object of mass M and radius R, the nonvanishing components of the Riemann tensor are $\sim GM/R^3$ and $x^k \sim \bar{r} \sim a_0 n^2$, where a_0 is the Bohr radius. Thus the order of magnitude of the energy shift is, to first order, $\Delta E \sim (GM\mu/R^3)a_0^2 n^4 = (4\pi/3)G\bar{\rho}\mu a_0^2 n^4$. It was first shown by Parker that, in order for the energy shift of a hydrogen atom in a tightly bound state at the horizon of a black hole to be of the same order as the Lamb shift (4.4×10^{-6} eV), the Schwarzschild radius should be $\sim 10^{-3}$ cm, typical of unobserved cosmological black holes [1].

The situation changes if, instead of concentrating on the first few tightly bound states of hydrogenlike atoms, one considers highly excited ($n \gg 1$), loosely bound ones, commonly referred to as Rydberg states [4]. Since the average distance of the electron from the nucleus is macroscopic in these cases ($\bar{r} \gg a_0$), the electron is much more sensitive to small external perturbations. Since several radio and optical lines from Rydberg atoms with principal quantum number up to $n \sim 350$ have been detected in various astrophysical environments (see, for instance, [5]), it is interesting to see how the above conclusions are modified in this case and if any observations of this process are possible. By using the above expression it can be seen that, for this gravitational tidal shift to be even just a small fraction of 1 μeV, it is necessary that n be of the order of or larger than $\sim 10^3$–10^4.

It is straightforward to verify, again from Eq. (1),

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that, for $n \sim 10^4$, the average Coulomb potential energy $\overline{V}_{\rm Coul} \sim -Ze^2/\overline{r} \sim -Ze^2/a_0 n^2$ also falls in the μeV range. This implies that, for values of n and of the Riemann tensor components for which the effect might be measurable, the perturbation induced by the gravitational field is a non-negligible fraction of the distance of the typical energy level from the continuum. This has three major consequences. The first is that, under these circumstances, the frequency shifts of recombination lines produced by bound-bound $(n+1)P \to nS$ transitions with $n \gg 1$ will be appreciable. The second consequence is that the probability of electron tunneling (*gravitational* field ionization) will be comparatively large, even though not necessarily important. Finally, fine structure effects, which scale as $1/n^2$ of the eigenenergy, can be neglected.

The details of the effects of a tidal perturbation on the Coulomb Hamiltonian are of course determined by the distribution of mass and energy via the particular field equations used. However, we can use the trace-free property of the Weyl conformal tensor [6] to draw some general conclusions about the quadratic form $H^{\rm tidal}_{\rm NR} = (1/2)\,\mu R_{0l0m}\, x^l x^m$ in Eq. (1) (we shall assume that form to be diagonal [1]).

Since we are working to first order in the metric expansion in series of the Riemann tensor, and since the metric is static, the trace of the Weyl tensor in vacuum is simply given by $R_{0101} + R_{0202} + R_{0303} = 0$ (if the adopted field equations yield $R = 0$, $R_{\alpha\beta} = 0$ in vacuum).

Thus at least one of three spatial elements of the diagonalized Riemann tensor must be negative in sign. This is of course well known in the spherically symmetric case, where we have [7] $R_{0101} = R_{0202} = GM/R^3$, $R_{0303} = -2GM/R^3$ if the z axis is oriented radially. The physical interpretation of this result is that it is never possible to produce a tidal field the only effect of which is to *stretch* (or to *squeeze*) the wave function in all spatial directions; rather, the atom is generally stretched in some directions and squeezed in others.

Since the probability of the electron tunneling out of the Coulomb potential well in at least one direction is nonvanishing, the overall spectrum is unbound, in qualitative analogy with the Stark effect. Under these circumstances stationary states do not rigorously exist, but are replaced by finite-width quasistationary resonances where the electron may spend an average time inversely proportional to the strength of the perturbation. Under these circumstances, any perturbative approach to the real part of the complex eigenvalues will only yield asymptotic series [8].

The expansion of the metric to first order in the Riemann tensor makes the problem separable (a reduced mass can be introduced), but not integrable. This is so because the Hamiltonian at Eq. (1) describes a reduced three-body problem (any effect of the electron on the gravitational source is neglected), which is not integrable even in classical mechanics. Details about all the mathematical methods employed are provided in a forthcoming publication ([4], and references therein).

The first-order corrections $E^{(1)}_{nS} = \langle \psi^{(0)}_{nS} | H^{\rm tidal}_{\rm NR} | \psi^{(0)}_{nS} \rangle$ for the nS states are

$$E^{(1)}_{nS} = \int_V \psi^{(0)}_{nS}(\mathbf{r})\, \frac{1}{2}\mu\, R_{0l0m}\, x^l x^m\, \psi^{(0)*}_{nS}(\mathbf{r})\, dV\,, \qquad (2)$$

where $\psi^{(0)}_{nS}(\mathbf{r})$ are the unperturbed wave functions. The calculation proceeds by performing a suitable change of variable in Eq. (2) and by evaluating integrals of the kind $\int_0^\infty \rho^4 e^{-\rho} L^k_{n-1}(\rho) L^k_{n-1}(\rho)\, d\rho$. This can be done by iteratively using the orthogonality property of the associated Laguerre polynomials [9]. A direct calculation shows that [10]

$$E^{(1)}_{nS} = \frac{\hbar^4}{12 Z^2 e^4 \mu} n^2 (5n^2 + 1) R_{00}\,, \qquad (3)$$

where R_{00} is the time-time component of the Ricci tensor. A similar calculation for the $l = 1$ states requires the diagonalization of the $\langle \psi^{(0)}_{nPm} | \frac{1}{2}\mu\, R_{0i0j}\, x^i x^j\, | \psi^{(0)}_{nPm'} \rangle$ 3×3 matrix and the use of the same above integrals involving the associated Laguerre polynomials. A direct calculation yields

$$E^{(1)}_{nP,i} = \frac{\hbar^4}{4 Z^2 e^4 \mu} n^2 (n^2 - 1)(R_{00} + 2R_{0i0i})\,, \qquad (4)$$

where $i = 1, 2, 3$ corresponds to the P_x, P_y, and P_z orbitals, respectively. The $E^{(1)}_{1S}$, $E^{(1)}_{2S}$, and $E^{(1)}_{2P}$ shifts given by the above equations correspond to those previously appearing in the literature [1].

The first-order frequency shifts of recombination lines for $(n+1)P \to nS$ transitions, commonly referred to as $\mathrm{H}n\alpha$ transitions ($\Delta n = 1$) are, in a spherical symmetry,

$$\Delta\nu^{\mathrm{H}n\alpha}_x = \Delta\nu^{\mathrm{H}n\alpha}_y = \frac{1}{3}\frac{\hbar^3 G}{Z^2 e^4 \mu}\, n(n+2)(n+1)^2\, \overline{\rho} = 3.38 \times 10^{-10}\, n(n+2)(n+1)^2\, \overline{\rho}_{14}\, \mathrm{Hz}\,, \qquad (5a)$$

$$\Delta\nu^{\mathrm{H}n\alpha}_z = -\frac{2}{3}\frac{\hbar^3 G}{Z^2 e^4 \mu}\, n(n+2)(n+1)^2\, \overline{\rho} = -6.76 \times 10^{-10}\, n(n+2)(n+1)^2\, \overline{\rho}_{14}\, \mathrm{Hz}\,, \qquad (5b)$$

where $\overline{\rho}_{14}$ is the average density expressed in units of 10^{14} g/cm^3.

In a spherical symmetry the energy shifts predicted by Eq. (3) vanish, and one would need to calculate corrections of order higher than the first in the Riemann tensor. This is of course incorrect within the framework of Eq. (1), which is valid only to first order. However, for the purpose of discussing some interesting orders of magnitude and of illustrating the general procedure, a calculation is carried out to obtain the second-order correction to the energy of the $2S$ state in the assumption that the Hamiltonian of Eq. (1) is exact. The calculation, which makes use of a no-wave-function procedure developed by Fernándes and Castro [11] yields

$$E_{1S}^{(2)} = -\frac{15}{2}\frac{\hbar^{10}}{\mu^3 e^{12}}(G\overline{\rho})^2 . \tag{6}$$

There is no guarantee that expressions for higher-order corrections for any n in closed form exist. Since high-order perturbative calculations for states with $n \gg 1$ are made quite difficult by the presence of large basis sets, the use of semiclassical Einstein-Brillouin-Keller (EBK) quantization methods is advised. Here we shall just present an elementary version of such an approach.

If we restrict ourselves to circular orbits in the x-y plane and by assuming the presence of a magnetic field in z (radial) direction, the equation of motion can be easily shown to be

$$\frac{\mu v^2}{\rho} = \frac{eBv}{c} + \frac{Ze^2}{\rho^2} + \mu R_{0101}\,\rho , \tag{7}$$

where $\rho^2 = x^2 + y^2$. By imposing the EBK quantization condition, and by eliminating the speed v, one obtains, after neglecting higher-order terms in the Riemann tensor,

$$\frac{\rho}{a_0} + \frac{1}{4}\left(\frac{\rho}{R_L}\right)^4 = n^2 , \tag{8}$$

where a_0 is the Bohr radius, and R_L is a *generalized* Landau radius defined as

$$R_L \equiv \left(\frac{e^2B^2}{c^2\hbar^2} + \frac{4\mu^2 R_{0101}}{\hbar^2}\right)^{-1/4} . \tag{9}$$

In curved space-time this quantity is in general different from the flat-space Landau radius $R_{L0} = (c\hbar/eB)^{1/2}$. If $B = 0$, the generalized Landau radius reduces to

$$R_g \equiv \left(\frac{\hbar^2}{4\mu^2 R_{0101}}\right)^{1/4} . \tag{10}$$

This *tidal gravitational* radius can be interpreted as the minimum size of the region in which an electron constrained to move in the x-y plane can be confined by tidal forces according to the uncertainty principle [4]. By solving Eq. (8) to first order in this case, one obtains the same first-order corrections to the energy as Eq. (4) for $n \gg 1$, to within at the most a factor of 2. This semiclassical model allows one to express the first-order energy shifts in terms of the fourth power of the ratio of the Bohr radius to the tidal Landau radius, $(a_0/R_{L0})^4$. In the case of intense magnetic fields, the spectrum of hydrogenlike atoms displays the characteristic quasi Landau modulations spaced by approximately $1.5\hbar\omega_c$, where ω_c is the cyclotron frequency [12]. By again using a semiclassical approach, it is possible to show that the introduction of space-time curvature causes a change in such a modulation spacing of the order of $\sim (R_{L0}/R_g)^4$. In the magnetospheres of pulsars $B \sim 10^{12}$ G, which makes $(R_{L0}/R_g)^4 \sim 10^{-32}$. This allows one to conclude that tidal gravitational perturbations on the structure of the quasi Landau spectrum of typical neutron stars are impossible to observe. For fields $B \sim 10^3$ G, the effect is still $\sim 10^{-13}$. It should be mentioned, however, that there is still considerable uncertainty as to the possibility of the existence of such low field pulsars, and that radiation from such objects could be observed only for high rotational speeds of the neutron star [13].

Of course, in order for any observation to be possible even only in principle, line broadening processes, such as natural, Doppler, and collisional broadening, and Stark interaction with neighboring atoms, must be sufficiently depressed (we neglect possible line profile broadening due to rotation for the moment).

The multidimensional semiclassical (EKB) quantization in the general case cannot yield information concerning the tidal field ionization probability [14]. However, some orders of magnitude can be obtained by studying spherically symmetric potentials of the kind $U(r) = -Ze^2/r - (GM\mu/R^3)\,r^2$. In this case the problem is separable and WKB quantization can be performed directly. By approximating the shape of the effective potential close to the maximum $U_{\max}$ as a parabola, one obtains that the transmission coefficient is, with exponential precision,

$$T = \frac{1}{1 + \exp\left[\dfrac{E - U_{\max}}{\hbar}\dfrac{1}{\sqrt{G\overline{\rho}}}\right]} . \tag{11}$$

This equation shows that the transmission coefficient depends exponentially on the ratio of the two characteristic times of the problem, that is, the period of recombination of the transitions, and the dynamical free-fall time of the configuration. In the case of a $\overline{\rho} = 10^{14}$ g/cm^3 object, the transmission coefficient is appreciable only for states within $\sim 10^{-12}$ eV of the continuum. This shows that line broadening due to the limited lifetimes of the states will be negligible under most circumstances.

Since both the radiative and resonance lifetimes are comparatively large, the only significant effects are Doppler, collision, and electron pressure (Stark) broadening. In order for the tidal energy shifts to be observable, line broadening processes must not cause the H$n\alpha$ lines to merge with one another and into the continuum. Well-known elementary considerations yield the order of magnitude of the Doppler and of the collisional broadenings as $\Delta\nu_D \sim (e^4\mu/2\pi\hbar^3 c)(kT/m_H)^{1/2}$ and $\Delta\nu_c \sim (\hbar^4/2\mu^2 e^4)(3kT/m_H)^{1/2}\, n^4\mathcal{N}$, respectively (we have specialized to the case of atomic hydrogen). An approximate expression for the electron pressure broadening was obtained by Griem [5]. By using these three expessions and the approximate equation for the distance between two contiguous H$n\alpha$ lines, one obtains three constraints on the temperature T, the total number density $\mathcal{N}$, and the electron number density $\mathcal{N}_e$ of the media involved depending on the value of the principal quantum number n considered [4]. These constraints are, respectively,

$$T \ll \frac{1.1\times10^{13}}{n^2}\,\mathrm{K}\,, \tag{12a}$$

$$\mathcal{N} \ll \frac{9.0\times10^{28}}{T^{1/2}}\frac{1}{n^8}\,\mathrm{cm}^{-3}\,, \tag{12b}$$

$$\mathcal{N}_e \ll \frac{1.0\times10^{14}}{n^8}\frac{T}{1/2+\ln(6.6\times10^{-6}Tn)}\,\mathrm{cm}^{-3}\,. \tag{12c}$$

By assuming a turbulent velocity of the medium $v_{\rm tur} \sim 0.1$ km/s, one obtains a turbulent Doppler broadening $(\Delta\nu/\nu)_{\rm tur} \sim 3\times10^{-6}$, which sets an order-of-magnitude limit to the smallest observable frequency shift. From Eqs. (5a) and (5b) it results that, for instance, $(\Delta\nu/\nu)_x^{\rm tidal} \approx (2\pi/3)(\hbar^6 G/Z^4 e^8 \mu^2)\, n^7\, \overline{\rho}$, for $n \gg 1$. Thus the above condition is satisfied for $n \approx 900$ or larger.

The Doppler broadening constraint could be marginally satisfied at typical surface temperatures in the 10^6 K range, since Eq. (12a) demands that $T \ll 10^7$ K for the indicated values of n. In the absence of electromagnetic fields, the values of both the ion and electron densities above the surface would be negligible at distances larger than the pressure scale height $h \sim kTR^2/m_H GM \sim 1$ cm.

It is also important to point out that, since the recombination radiation is received from an atmosphere surrounding the object, the lines would not display a shift but rather a broadening pattern, reflecting the range of values of the Riemann tensor components in the region.

One must conclude from the results presented in this Letter that, unlike the atoms in their first few excited states studied by Parker, Rydberg atoms in very high energy levels could in fact be used as probes of space-time properties in realistic astrophysical objects, even though not in the strong mixing or quasi Landau regimes. Radio-quiet neutron stars which may have been detected only through their gravitational interaction with an orbiting companion could represent potential observational targets [13]. In this case, however, the lack of radio emission from the Goldreich-Julian mechanism [15] may make the detection of radiation from these objects problematic.

Research on the behavior of one-electron atoms in metrics other than that of Schwarzschild, such as, for instance, in the case of rapidly rotating dense objects, and on the case of nonradial magnetic fields, will be reported in a forthcoming publication [4].

If objects where tidal gravitational perturbations on atoms can be measured are identified, one will be able to approach the study of the gravitational effects on Rydberg atomic systems from the observational point of view. In perspective, this might represent a new observational tool for the spectroscopic determination of the average density of neutron starlike objects.

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[1] L. Parker, Phys. Rev. Lett. **44**, 1559 (1980); Phys. Rev. D **22**, 1922 (1980).

[2] W. Greiner, B. Müller, and J. Rafelski, *Quantum Electrodynamics of Strong Fields* (Springer-Verlag, Berlin, Heidelberg, 1985), p. 550.

[3] F. K. Manasse and C. W. Misner, J. Math. Phys. **4**, 735 (1963); C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973), p. 332.

[4] F. Pinto, Bull. Am. Astron. Soc. **24**, No. 4 (1992) (abstract); (to be published).

[5] H. R. Griem, Astrophys. J. **148**, 547 (1967); M. Brocklehurst and M. F. Seaton, Mon. Not. R. Astron. Soc. **157**, 179 (1972); H. R. Griem, *Spectral Line Broadening by Plasmas* (Academic, New York, 1974), p. 257; A. Dalgarno, in *Rydberg States of Atoms and Molecules*, edited by R. F. Stebbings and F. B. Dunning (Cambridge Univ. Press, Cambridge, 1983), p. 1.

[6] Misner, Thorne, and Wheeler, *Gravitation* (Ref. [3]), pp. 325 and 327.

[7] Misner, Thorne, and Wheeler, *Gravitation* (Ref. [3]), p. 37.

[8] L. D. Landau and E. M. Lifshitz, *Quantum Mechanics* (Addison-Wesley, Reading, MA, 1958); R. J. Damburg and V. V. Kolosov, in *Rydberg States of Atoms and Molecules* (Ref. [5]), p. 31; G. A. Arteca, F. M. Fernándes, and E. A. Castro, *Large Order Perturbation Theory and Summation Methods in Quantum Mechanics* (Springer-Verlag, Berlin, Heidelberg, 1990), p. 322.

[9] G. Arfken, *Mathematical Methods for Physicists* (Academic, Orlando, FL, 1985), pp. 726 and 728.

[10] This result can also readily be obtained from the isotropic average of the perturbation in Eq. (1) by exploiting the isotropic property of the S waves and the well-known value of $\langle r^2\rangle$. It is by no coincidence, however, that, in general, the calculation of $\langle r^k\rangle$ requires the calculation of the same integrals involving the associated Laguerre polynomials as discussed in the text [8].

[11] F. M. Fernándes and E. A. Castro, Int. J. Quantum Chem. **26**, 497 (1984); **28**, 603 (1985); Arteca, Fernándes, and Castro, *Large Order Perturbation Theory and Summation Methods in Quantum Mechanics* (Ref.

[8]), p. 63.
[12] R. F. O'Connell, Astrophys. J. **187**, 275 (1974); Arteca, Fernándes, and Castro, *Large Order Perturbation Theory and Summation Methods in Quantum Mechanics* (Ref. [8]), p. 229; R. H. Garstang, Rep. Prog. Phys. **40**, 105 (1977); M. Gutzwiller, *Chaos in Classical and Quantum Mechanics* (Springer-Verlag, New York, 1990).
[13] G. Chanmugam, Annu. Rev. Astron. Astrophys. **30**, 143 (1992), and references therein.
[14] I. C. Percival, Adv. Chem. Phys. **36**, 1 (1977).
[15] P. Goldreich and W. H. Julian, Astrophys. J. **157**, 869 (1969).