Generalized Hard-Core Fermions in One Dimension: An Exactly Solvable Luttinger Liquid

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A model for interacting spinless fermions in one dimension is presented and solved exactly. The solution is shown to be a Luttinger liquid with charge density wave instabilities for well characterized filling factors. Under very plausible assumptions, the exact solution is extended to the spin case with the usual charge-spin separation into two different Luttinger liquids. Required technicalities are almost trivial but the solution is very rich: Periodicity of instabilities and correlation exponents can be chosen at will.

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One-dimensional (1D) models provide examples of truly interacting systems for which exact solutions can be constructed [1]. Beyond being relevant for an increasing number of quasi-one-dimensional materials [2], they have an intrinsic interest as models of *metals* which are not Landau's Fermi liquids [3,4]. Renewed interest in their study has been prompted by Anderson's proposal [5] that normal state properties of high T_c superconductors are closer to the known behavior of 1D models than to the customary Fermi liquid picture.

Although exact information about interacting fermion systems in 1D has been with us for a long time, recognition of an underlying universal structure for all of them is rather recent. According to Haldane [6], all spinless, gapless, and interacting 1D fermion systems are Luttinger liquids (LL): They share universal features in their low energy physics corresponding to the exactly solvable Luttinger model [7]. This is the quantum 1D version of the classical 2D Gaussian model [8]: a critical system with continuously varying exponents. Elementary excitations are phononlike density fluctuations governed by a harmonic Hamiltonian. If one includes global particle and current fluctuations, the diagonalized Luttinger model is written as [6]

$$\mathcal{H}_L = v_s \sum_q |q| b_q^\dagger b_q + \frac{\pi}{2L}[v_N (N-N_o)^2 + v_J J^2] , \quad (1)$$

where the first term is the bosonlike content of density fluctuations of wave vector g. $N(N_o)$, J, and L are particle number, current number, and system length, respectively. A key aspect of the LL concept is the following scaling relation between the three velocities of the spectrum, $v_J v_N = v_S^2$. Its importance resides in that it provides us with a dimensionless number [6], the *exponent parameter* $e^{-2\phi} = v_N/v_S$, that fully characterizes criticality, and whose departure from one is a measure of the effective strength of interactions.

This proposed universal picture is confirmed by the known Bethe ansatz (BA) solvable models. In spite of their physical opacity and the difficulties in extracting relevant information, there is enough evidence that they are indeed LL [6,9–11]. Unfortunately, these difficulties make BA solutions close to useless if one wishes to take them as starting points for the various interesting questions related to the LL concept (2D coupling stability, transport and disorder effects, etc.).

In this Letter we solve exactly a class of spinless (and under plausible assumptions, with spin) fermion models which fit into the classification of LL. The required technicalities are kept to a bare minimum while the solution can exhibit a very rich structure, both virtues in striking contrast with standard BA models. Let us consider N spinless fermions running through a chain of length L and governed by the following self-explanatory Hamiltonian:

$$\mathcal{H} = -t\sum_i (c_i^\dagger c_{i+1} + \text{H.c.}) + \lambda \sum_{i,m} U_m n_i n_{i+m} . \quad (2)$$

The interaction is repulsive, short range ($U_m = 0$, if $m > p$), and otherwise arbitrary, except for the condition $U_m < (U_{m+1} + U_{m-1})/2$. The exact solution will be found in the limit $\lambda \to \infty$. The reason for the peculiar condition on the potential energy will become clear soon. For the moment, let us present the simple strategy that allows an exact solution. In the specified limit, the interaction term (static energy) becomes dominant: States with different static energies do not couple to each other. Therefore, the solution implies the identification of all the states that share a given static energy and see what the kinetic energy term does on this manifold.

The lowest static energy is zero: Fermions do not overlap their hard core. They move freely, the interaction becoming the constraint that the minimum mutual distance is $p+1$ lattice units. The presence of a particle is just a restriction on the available space for the others, and one can map this problem onto one in which the constraint has disappeared: free fermions with the same hopping parameter [12]. The mapping is a particle-dependent length rescaling where the position of a particle in the real system is moved to the left a quantity proportional to the total space stolen by all the other particles to its left. The rule is $\tilde{i} = i - N_i p$, where i ($\tilde{i}$) represents a position in the original (mapped) lattice, and N_i is the total number of fermions between the origin and site i. If the original Hamiltonian has a density $Q = N/L$, it is straightforward to see that the mapped Hamiltonian corresponds to free fermions in a lattice shrunk to

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$\tilde{L}=L(1-Qp)$ and with a Fermi vector increased to $\tilde{k}_F=k_F/(1-Qp)$.

The mapped model is trivially solvable and, therefore, so is the real problem in the subspace of zero static energy. Concerning LL behavior, velocities are easily obtained from the total energy of the system E_T:

$$v_N=\frac{L}{\pi}\frac{\partial^2 E_T}{\partial N^2}=\frac{2t}{(1-Qp)^3}\sin\left(\frac{Q\pi}{1-Qp}\right). \quad (3)$$

v_J is obtained from the change in energy upon piercing a flux through the chain. This amounts to changing the hopping term from t to $te^{i\theta}$, and

$$v_J=\frac{\pi}{L}\frac{\partial^2 E_T}{\partial\theta^2}=2t(1-Qp)\sin\left(\frac{Q\pi}{1-Qp}\right). \quad (4)$$

v_S is equally simple: The mapped free fermion Hamiltonian is the simplest LL and a density fluctuation mode with frequency $\tilde{v}_S\tilde{q}$ and wave vector $\tilde{q}$ corresponds to a density fluctuation mode in the real system with equal frequency and $q=\tilde{q}(1-Qp)$; therefore,

$$v_S=\frac{\tilde{v}_S}{1-Qp}=\frac{2t}{1-Qp}\sin\left(\frac{Q\pi}{1-Qp}\right), \quad (5)$$

with $\tilde{v}_S$ the sound velocity of the free case. The three velocities satisfy the LL scaling relation, with $e^{-2\phi}=(1-Qp)^{-2}$. Notice the effort in obtaining these values from their physical definition without assuming any *a priori* relation between them. The satisfaction of the scaling relation is the manifestation of the LL nature of this problem.

This solution applies to any hard-core Hamiltonian in the zero static energy sector, irrespective of details inside the hard core, and survives up to a critical density $Q_{c1}=1/(p+1)$. At this point, velocities go to zero and a charge density wave (CDW) instability develops. It is the lowest order commensurate structure that can appear and corresponds to a closed packing of particles in the zero static energy sector. An (infinite) gap in excitations appears because any attempt to shake this closed packed structure makes particles overlap, with the corresponding penalty in energy.

Changing (by the corresponding infinite amount) the chemical potential, we can go beyond this commensurate structure. Imagine compressing the commensurate structure by one lattice unit. Two nearest neighbor fermions reduce their distance by one, creating a domain wall (DW), or soliton, between two pieces of the commensurate structure. This object can be anywhere, and the corresponding sector is degenerate. The kinetic energy part moves this object as if it were a free fermion running through the lattice with elementary hops of $p+1$ lattice units. Suppose we compress the system one further lattice unit. Two things can happen: Either a new similar DW is created or the original one accommodates the new compression. The choice of the potential mode guarantees that the latter possibility never happens: It would imply an additional infinite penalty in energy. Other choices would lead to phase segregation between different commensurate phases, something we want to avoid. Now the physical picture is clear: Above the critical filling Q_{c1}, the excess density is accommodated by DW's of this commensurate structure. These objects are fermions (no crossing of world lines), moving freely with $p+1$ length hops, but constrained to have a minimum mutual distance of p lattice units, as can be easily checked. This constraint is similar to the one already handled, but with opposite sign: The interacting gas of DW's can be mapped onto a free one with a DW's density dependent lattice rescaling (expansion in this case). Following the same procedure as before, the problem is exactly solved, now with a further intermediate step: from bare particles to interacting DW's later mapped onto free DW's. The velocities are evaluated and shown to satisfy the scaling relation, providing the exponent parameter and a complete characterization of this LL.

This LL survives until a new critical concentration $Q_{c2}=1/p$. At this point we have a rigid structure of DW's which is the next commensurate CDW instability: Attempts to shake it cost an infinite amount of energy. Further compression creates DW's of the new reference commmensurate structure: These are fermions moving with elementary hops of p lattice units and with closest distance of $p-1$ lattice units, etc. Now the pattern for a complete solution is clear and can be repeated until the CDW structure corresponding to $Q_M=1/M$, where M is the lowest integer satisfying $2\leq M>p/2$. For $Q\in[1-Q_M,1]$, electron-hole symmetry completes the task. For band fillings in the window $Q\in[Q_M,1-Q_M]$, DW's structure and interactions become more involved and the previous simple procedure cannot be pursued to obtain an exact solution [13]. Notice that for $p<4$, $Q_M=\frac{1}{2}$, and such a window does not exist. Also, we could have avoided this problem (at the expense of electron-hole symmetry) by deciding that only contiguous fermions experience mutual interactions [13] (an n-body coupling in disguise), but, for simplicity, we stick to the original definition [Eq. (2)] without further qualifications.

Therefore, we can summarize the exact solution as follows ($0\leq Q\leq Q_M$, with the corresponding electron-hole symmetric counterpart).

(i) The system shows long range commensurate CDW at fractional fillings $Q_c=1/m$ with $m=p+1,p,\ldots,M$. Particles are equally spaced (superperiod is m lattice units).

(ii) Between commensurate structures, the system is a LL liquid, with interacting (constrained) DW's of the low density commensurate structure as basic dynamical objects (bare particles can be considered DW's of the *vacuum* commensurate structure). The LL behavior is characterized by (just one velocity)

$$v_S = \frac{2t}{1-Qp}\sin\left[\frac{Q\pi}{1-Qp}\right],\quad e^{-2\phi}=(1-Qp)^{-2} \qquad (6)$$

if $Q \le 1/(p+1)$, and

$$v_S = \frac{2t}{Q}|\sin(\pi/Q)|,\quad e^{-2\phi}=Q^{-2}, \qquad (7)$$

otherwise. In Fig. 1 we show the relevant LL information for the model with $p=3$.

The following comments are appropriate. The solution is complete in the sense that all eigenstates can be obtained exactly (free fermions in the mapped system), though only the low energy part is relevant for the LL picture. We believe the qualification of this problem as a LL is already safe. However, from a purist point of view, one would desire an independent verification from correlations and not only from the spectral velocities, but calculating correlations is not easy. The origin of the difficulty reflects a physical feature expected for any LL: If one imagines, for instance, introducing a particle in the real system, the corresponding action in the mapped problem implies doing so at a rescaled position and, *in addition,* a major nonlocal rearrangement of the whole system. In spite of these difficulties, we have succeeded in calculating, for instance, the single particle correlations with the only information contained in the mapping without invoking the LL concept. The results, of course, agree with expectations [13]. Nevertheless, an independent, correlation-based proof of the correctness of the LL picture for this problem is possible here. Elementary geometrical reasoning [12] tells us that, for low amplitude, long wavelength fluctuations, there is always a simple relation between original and mapped problems for the quantities $\langle(\Delta\tilde{N}_{\tilde{l}})^2\rangle$ and $\langle(\Delta N_l)^2\rangle$, where ΔN_l is the particle number fluctuation in a segment of length l, with the corresponding definition in the mapped version. For example, in the zero static energy sector, $(1-Qp)^2 \times\langle(\Delta\tilde{N}_{\tilde{l}})^2\rangle=\langle(\Delta N_l)^2\rangle$, with straightforward generalization to other sectors. From the knowledge of this quantity in the free mapped case, one immediately obtains the exponent parameter $e^{-2\phi}$. As expected, the results are in complete agreement with the evaluation from velocities. Notice that, from the Gaussian model point of view, just one correlation is needed to characterize criticality. In this sense, the previous calculation can be considered enough.

Unlike solved BA models, the exponent parameter can be changed arbitrarily. In particular, we can be beyond the point at which the occupation number shows a singularity in the first derivative at the Fermi surface: $\cosh(2\phi)=2$. Thus, this model provides us with a natural system for the investigation of LL stability versus 2D coupling [14]. Notice how the exponent parameter $e^{-2\phi}$ decreases from the first commensurate structure to Q_M as a consequence of the attractive interaction between DW's: More DW's implies more available space in the mapped free problem.

The periodicity of the lowest order CDW instability, $m=p+1$ lattice units, depends on the range of interactions in the manner expected from the theory of commensurate-incommensurate transitions [15]. Also, from this point of view, the CDW instabilities are consistent with expectations: If we are given the expression of the exponent parameter without explicit reference to instabilities, scaling arguments imply that the LL is unstable to sine-Gordon perturbations only for exactly the periodicities of the obtained CDW phases. The same scaling arguments [15] imply that the exponent parameter tends to the value $e^{-2\phi}=m$ while approaching the density of a CDW with superperiodicity m, as it obviously happens in the exact solution. A further test of consistency is provided by the infinite coupling limit of the *XXZ* model, the $p=1$ case of our Hamiltonian, whose BA solvable equations lead to the same results of our much more simple solution.

As mentioned before, quasifree solitons are the dynamical objects slightly above any commensurate structure. Also, quasifree antisolitons (elementary dilations of the reference commensurate structure) are the dynamical units slightly below any commensurate structure. These objects carry a charge excess (defect) of $1/m$, in original particle units. They offer an easy visualization for the breakdown of Fermi liquid quasiparticle behavior, similar to the spin-charge separation of spin cases: The bare particle no longer is the basic dynamical unit, apparently a recurrent situation in LL problems. With this picture of charge division, for instance, recent results concerning the behavior of a perfect LL liquid in the presence of an electric field [16] can be naturally explained in our model.

We now show that under very plausible assumptions, the solution presented so far for spinless fermions is also an exact solution in the spin case. Consider spin-$\frac{1}{2}$ fermions governed by the same $\mathcal{H}$, supplemented with an on-site interaction U_0 chosen to comply, for the same reasons, with the previous requirement: $U_0 > 2U_1 - U_2$. In the infinite λ limit, we have an extreme case of charge-spin separation: Every previously considered ei-

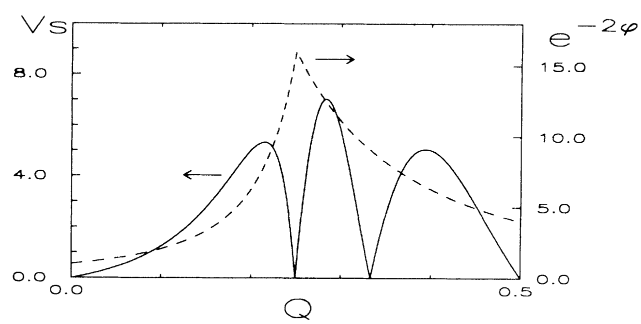

FIG. 1. Sound velocity and exponent parameter vs particle density for the spinless Hamiltonian with $p=3$.

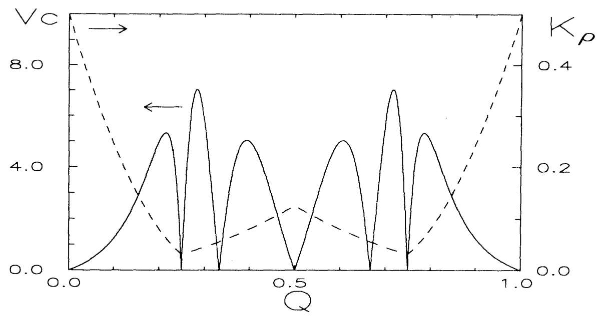

FIG. 2. As in Fig. 1 for the spin case.

genstate (charge eigenstate) displays now the full degeneracy of spin degrees of freedom. To solve the problem, all that remains is to analyze how, in the large though finite λ limit, the residual spin interaction disposes of this degeneracy. This will lead to a spin Hamiltonian for every charge eigenstate with the following physical requirements: spin isotropy, extreme locality (it is mediated by very energetic short-lived charge fluctuations), and, on obvious grounds, antiferromagnetic sign. Thus, we are forced to conclude that every charge eigenstate organizes its spin degeneracy according to the Heisenberg antiferromagnetic Hamiltonian: $\mathcal{H}_{\text{spin}}=J_{\text{eff}}\sum_i \mathbf{s}_i\cdot\mathbf{s}_{i+1}$, with J_{eff} vanishing as $O(t^2/\lambda)$. Notice that index i describes order and not lattice position: Spin fluctuations float on top of the charge fluctuations. This structure is precisely obtained in the BA solution of the infinite U [17] standard Hubbard model [18], the simplest particular case of our family of Hamiltonians with the charge sector corresponding to free spinless fermions, $p=0$. With this in mind, the spin case is already solved, and amounts to an unfolding of the corresponding spinless solution (spinless full occupation corresponds to spin-half occupation). Adopting a customary notation [9], the exact solution is characterized by $v_c=v_s$, $K_\rho=\frac{1}{2}e^{2\phi}$. Notice that knowledge of K_ρ is enough to obtain all correlation exponents [9]. Figure 2 summarizes the spin solution corresponding to the spinless case of Fig. 1, for densities between zero and half filling (the rest from electron-hole conjugation).

In summary, we have presented an exactly solvable model for spinless fermions in 1D. The solution shows the system to be a LL with CDW instabilities at well defined fractional fillings. Modulo very plausible assumptions, the exact solution is extended to the spin case, exhibiting the usual charge-spin separation into two LL. Unlike BA models, the technicalities are almost trivial while the solution exhibits a rich structure: For instance, the periodicity of commensurate phases and the exponents of the LL can be changed at will by modifying the range of interactions. We believe these properties make this model a natural choice in order to exemplify the universal nature of the LL picture with an exactly solvable case, and as a manageable starting point for further studies of generic LL properties.

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