Superposition of Nonlinear Plasma Waves

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We report results showing that spatially periodic Bernstein-Greene-Kruskal (BGK) waves, which are exact nonlinear traveling wave solutions of the Vlasov-Maxwell equations for collisionless plasmas, satisfy a nonlinear principle of superposition in the small amplitude limit. The analysis explicates the notion of superimposed BGK waves which, as recent numerical calculations suggest, is crucial in the proper description of the time-asymptotic state of a plasma when a large amplitude electrostatic wave undergoes nonlinear Landau damping.

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Many plasmas exhibit complex collective dynamical behavior on time scales over which the effects of discrete particle interactions are negligible. Examples occur, for instance, in interplanetary and astrophysical settings, as well as in the laboratory. The nonlinear Vlasov-Maxwell equations [1] of kinetic theory, which describe the physics of such "collisionless" plasmas, reduce for longitudinal electrostatic processes to the one-dimensional Vlasov-Poisson-Ampère equations,

$$\frac{\partial f_\alpha}{\partial t}+u\frac{\partial f_\alpha}{\partial x}+\frac{q_\alpha}{m_\alpha}E\frac{\partial f_\alpha}{\partial u}=0\,, \tag{1}$$

$$\frac{\partial E}{\partial x}=4\pi\sum_\alpha q_\alpha\int_{-\infty}^{\infty}du\,f_\alpha\,, \tag{2}$$

$$-\frac{\partial E}{\partial t}=4\pi\sum_\alpha q_\alpha\int_{-\infty}^{\infty}du\,u f_\alpha\,, \tag{3}$$

where $f_\alpha(x,u,t)$ is the distribution function for particle species α, $\alpha=1,2,\dots,N$, and $E(x,t)$ is the self-consistent longitudinal electric field. Distribution functions $F_\alpha(u)$ that yield vanishing charge and current densities characterize the spatially uniform zero-field equilibrium solutions of Eqs. (1)-(3), the so-called "Vlasov equilibria," which correspond to long-lived metastable configurations of the physical plasma. Much of our understanding of wave processes near any of these equilibrium states, including the well-known phenomenon of collisionless damping of small amplitude waves in a broad class of plasmas, follows from Landau's classic analysis [2] based upon Eqs. (1)-(3) linearized about an equilibrium $F_\alpha(u)$.

That the results of the linear analysis are fundamentally incomplete, however, is clear from the work of Bohm and Gross [3] who showed, in the context of the full nonlinear equations, that there exist traveling wave solutions of constant amplitude, i.e., undamped waves. For these exact nonlinear solutions the distribution functions depend on the coordinates x, u, t only through the conserved single-particle energy $\mathcal{E}_\alpha=m_\alpha(u-v)^2/2+q_\alpha\varphi(x-vt)$, where v is the wave velocity. Importantly, it is a distinctive feature of these waves, as opposed to those described by the linear theory, that some of the plasma particles are trapped within the traveling electrostatic potential wells. In fact, it is possible, as shown by Bernstein, Greene, and Kruskal [4], to choose the distributions of such trapped particles appropriately so as to create electrostatic plasma waves with an essentially arbitrary relationship between frequency ω and wave number k. For Bernstein-Greene-Kruskal (BGK) waves of small amplitude, however, the relationship between ω and k is less arbitrary, and is given in the limit of zero amplitude by the Vlasov dispersion relation [1,3,5]

$$1-\frac{4\pi}{k^2}\sum_\alpha\frac{q_\alpha^2}{m_\alpha}P\int du\frac{F_\alpha'(u)}{u-\omega/k}=0 \tag{4}$$

(*not* the Landau dispersion relation [2]), where P denotes the principle value, and the functions $F_\alpha(u)$ describe the equilibrium plasma state.

Since BGK waves are exact solutions of nonlinear equations they do not, of course, satisfy a principle of linear superposition. In fact, linear superposition does not obtain *even in the limit of zero amplitude,* since the fundamentally nonlinear phenomenon of particle trapping remains essential at all amplitudes. Nevertheless, the notion of superimposed BGK waves recently has been shown to be crucial in accurately describing the time-asymptotic states obtained in numerical simulations of plasma evolution from various simple initial conditions [6]. This numerical evidence strongly suggests, in fact, that there exists a nonlinear superposition principle for spatially periodic BGK waves of small amplitude, in which the electric fields superimpose linearly while the distribution functions combine in some more complicated way. Since superimposed wave states were utilized in recent experimental studies of dynamical chaos in the interaction of two constant amplitude electrostatic waves [7]. In this Letter we develop a principle of nonlinear superposition relevant to these observations [6,7], and give conditions on the wave amplitudes and the relative wave velocity under which it applies.

That BGK waves do not superimpose linearly even in the limit of zero amplitude can be seen as follows. Suppose that $(f_\alpha^{(1)},\varphi^{(1)})$ and $(f_\alpha^{(2)},\varphi^{(2)})$ are distribution functions and electric potentials corresponding to BGK

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waves with respective velocities v_i and v_2, where $h_a^{(i)} = f_a^{(i)} - F_a$ are the small deviations of the distribution functions from those of a Vlasov equilibrium F_a. The linearly superimposed state then is $f_a^L = F_a + h_a^{(1)} + h_a^{(2)}$, $\varphi^L = \varphi^{(1)} + \varphi^{(2)}$, in which the distribution functions are given by their equilibrium values plus the deviations $h_a^{(1)}$ and $h_a^{(2)}$ corresponding to the two separate waves. This linear superposition satisfies both the Poisson and Ampère equations, Eqs. (2) and (3), exactly by virtue of their linearity, while the Vlasov equation becomes

$$\left(\frac{d}{dt}\right)_L f_a^L = -\frac{q_a}{m_a}\left[\frac{\partial \varphi^{(1)}}{\partial x}\frac{\partial h_a^{(2)}}{\partial u} + \frac{\partial \varphi^{(2)}}{\partial x}\frac{\partial h_a^{(1)}}{\partial u}\right], \quad (5)$$

where $(d/dt)_L$ denotes the time derivative evaluated along particle trajectories in the superimposed field $\varphi^L(x,t)$. For waves with amplitudes that are of order ϵ, the deviations $h_a^{(i)}$ and potentials $\varphi^{(i)}$ are each of first order in that small parameter, as is the left-hand side of Eq. (5). That the right-hand side of Eq. (5) appears to be second order in ϵ suggests that (f_a^L, φ^L) is an approximate superimposed solution which becomes exact as $\epsilon \to 0$. However, since BGK waves trap particles even at small amplitudes, the single-wave distribution functions $f_a^{(i)}$ each must satisfy for all ϵ the condition $\partial f_a^{(i)}/\partial u|_{u=v_i} = 0$, or equivalently, $\partial h_a^{(i)}/\partial u|_{u=v_i} = -dF_a/du|_{u=v_i}$, in accordance with the formation of a small plateau at the wave velocity in the distribution functions. But this means that in the neighborhoods of both phase velocities v_1 and v_2, the right-hand side of Eq. (5) actually is only first order in ϵ, and thus remains important even as the wave amplitude become very small. Thus, the linear superposition fails due to particle trapping and the associated particular nonlinearity of the Vlasov equation. Nevertheless, below we construct, by a nonlinear rule for superposition, a two-wave solution that does not suffer from the above difficulty, i.e., that becomes exact in the limit $\epsilon \to 0$.

Consider the trajectories of single particles in the plasma. In the case of a uniformly translating wave $\varphi = \varphi(x - vt)$, the particle energies $\mathcal{E}_a = m_a(u-v)^2/2 + q_a\varphi(x - vt)$ are exact invariants of the motion. The existence of these simple invariants, in fact, gives the BGK method its power, since the Vlasov equation is solved exactly by $f_a(x,u,t) = g_a(\mathcal{E}_a)$ for any set of smooth, nonnegative functions g_a. The situation is far more complicated, however, if the field $\varphi(x,t)$ has nontrivial time dependence that cannot be transformed away by a simple shift in reference frame. In any such field (which corresponds, except in special cases, to a nonintegrable single-particle Hamiltonian) the energies $\mathcal{E}_a$ no longer are conserved, and accordingly $f_a(x,u,t) = g_a(\mathcal{E}_a)$ does not satisfy the Vlasov equation. The failure of the simple linear superposition (f_a^L, φ^L) is therefore not surprising, since the distribution functions are written in terms of quantities that are not invariant in the field φ^L. A two-wave solution can be developed correctly, however, by first finding invariant quantities for the two-wave field, and then using these true invariants to construct distribution functions. Thus we search for the proper generalizations of the two single-wave energy invariants $\mathcal{E}_a^{(i)} = m_a(u - v_i)^2/2 + q_a\varphi^{(i)}(x,t)$ to the two-wave case.

We consider the nearly integrable Hamiltonian system with Hamiltonian

$$H(x,p,t) = p^2/2m_a + q_a\varphi^{(1)}(x,t) + q_a\varphi^{(2)}(x,t) \quad (6)$$

for a charged particle in the field of two small amplitude electrostatic waves, $\varphi^{(1)}(x,t) = -\epsilon\bar{\varphi}_1\cos(k_1x - \omega_1 t)$ and $\varphi^{(2)}(x,t) = -\epsilon\bar{\varphi}_2\cos(k_2x - \omega_2 t)$, where $\bar{\varphi}_1, \bar{\varphi}_2 \sim 1$ and $\epsilon \ll 1$. For definiteness we consider $(v_1 = \omega_1/k_1) > (v_2 = \omega_2/k_2)$. For any nonintegrable Hamiltonian such as this it is impossible to find global invariants, i.e., quantities that are well behaved over the entire phase space. Nevertheless, approximate invariants, valid over restricted regions of phase space, can be developed via perturbation methods. Moreover, this approximate approach suffices for the present development since the stochastic regions occupy an area of the x-p phase plane that is exponentially small for small wave amplitudes, and the effects of using the approximate invariants over these regions is higher order and thus can be ignored in this limit.

For the Hamiltonian given by Eq. (6) it is straightforward to use perturbation theory to find quantities invariant through first order in the small parameter ϵ. A pair of first-order invariants which reduce for $\epsilon = 0$ to the single-wave energy invariants $\mathcal{E}_a^{(i)}$ are

$$\bar{\mathcal{E}}_a^{(i)} = \mathcal{E}_a^{(i)} + \epsilon q_a \frac{u - v_i}{u - v_j}\varphi^{(j)}(x,t), \quad i,j = 1,2, \text{ cyclic}. \quad (7)$$

These two invariants are equal through first order in ϵ along the time-dependent curve $u(x,t) = v_{\text{avg}} + (2\epsilon/m_a\delta v)\Delta\varphi(x,t)$, where $v_{\text{avg}} = (v_1 + v_2)/2$, $\delta v = v_1 - v_2$, and $\Delta\varphi(x,t) = \varphi^{(1)}(x,t) - \varphi^{(2)}(x,t)$. Although neither $\bar{\mathcal{E}}_a^{(1)}$ nor $\bar{\mathcal{E}}_a^{(2)}$ is a global first-order invariant by virtue of the singularities at the phase velocities v_2 and v_1, respectively, nevertheless, $\bar{\mathcal{E}}_a^{(1)}$ is well behaved for $u \geq u(x,t)$, as is $\bar{\mathcal{E}}_a^{(2)}$ for $u \leq u(x,t)$. Thus $\bar{\mathcal{E}}_a^{(1)}$ and $\bar{\mathcal{E}}_a^{(2)}$ taken together piecewise provide a pair of invariants which cover the entire x-u plane without singularities. These invariants are the key elements in the construction of the distribution functions for a two-wave state.

Figure 1 shows, for the Hamiltonian given by Eq. (6) with $q_a\bar{\varphi}_1/m_a = q_a\bar{\varphi}_2/m_a = 1$ and $\epsilon = 0.1$, in the specific case $k_1 = 2k$, $\omega_1 = 2\omega$, $k_2 = 3k$, and $\omega_2 = -3\omega$, numerically generated successive intersections of various particle trajectories with a Poincaré surface of section defined by stroboscopic sampling at times $t_n = nT$, $n = \ldots, -1, 0, 1, \ldots$, where $T = 2\pi/\omega$ is the period of the two-wave field. The particle velocity u is plotted vertically versus the variable ψ defined as $\psi = kx(\text{mod}2\pi)$. The closed invariant curves centered near the component wave velocities $v_1 = \omega/k$ and $v_2 = -\omega/k$ correspond to particles trapped by either of the two individual waves, while the

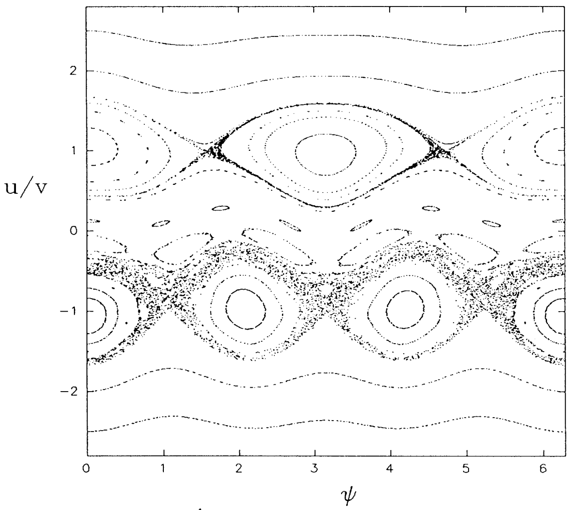

FIG. 1. The orbits of a Poincaré map constructed by stroboscopically sampling numerically generated particle trajectories in the two-wave field $\varphi(x,t)=-\epsilon\cos(2kx-2\omega t)-\epsilon\cos(3kx+3\omega t)$. Phase plane coordinates are (ψ,u) where $\psi=kx\times(\mathrm{mod}\,2\pi)$.

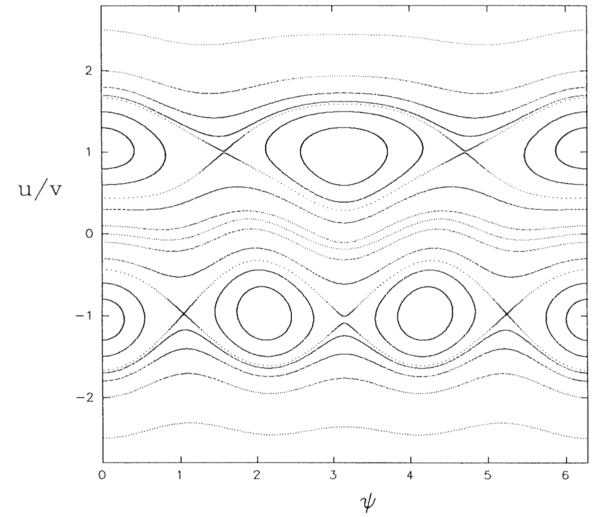

FIG. 2. Level curves of the first-order invariants $\bar{\mathcal{E}}_a^{(1)}$ and $\bar{\mathcal{E}}_a^{(2)}$, or equivalently, of the two-wave distribution function $f_a^{(+)}$, when evaluated on the Poincaré section. Curves were generated for $u\geq u(x,t)$ with $\bar{\mathcal{E}}_a^{(1)}$ and for $u<u(x,t)$ with $\bar{\mathcal{E}}_a^{(2)}$.

snakelike invariant curves at larger velocities correspond to untrapped particles. Figure 2 shows some of the level curves of the invariants $\bar{\mathcal{E}}_a^{(1)}$ and $\bar{\mathcal{E}}_a^{(2)}$ evaluated on the Poincaré surface; clearly, $\bar{\mathcal{E}}_a^{(1)}$ and $\bar{\mathcal{E}}_a^{(2)}$ capture the gross features of the particle dynamics, although they do not reflect the stochastic layers (dark areas corresponding to single trajectories) or the islands corresponding to higher-order resonances present in Fig. 1.

The approximate invariants $\bar{\mathcal{E}}_a^{(1)}$ and $\bar{\mathcal{E}}_a^{(2)}$ given in Eq. (7) can be used to construct distribution functions for a two-wave state. If $g_a^{(i)}$ are the BGK functions for the single waves such that $f_a^{(i)}(x,u,t)=g_a^{(i)}(\mathcal{E}_a^{(i)})$, then we define the distribution functions $f_a^{(+)}$ for the two-wave state as

$$f_a^{(+)}(x,u,t)=\begin{cases}g_a^{(1)}(\bar{\mathcal{E}}_a^{(1)}), & u\geq u(x,t),\\ g_a^{(2)}(\bar{\mathcal{E}}_a^{(2)}), & u\leq u(x,t).\end{cases} \tag{8}$$

That is, we use the function $g_a^{(1)}$ of the first wave and the associated invariant $\bar{\mathcal{E}}_a^{(1)}$ (not $\mathcal{E}_a^{(1)}$) above the curve $u(x,t)$, and the function $g_a^{(2)}$ of the second wave and $\bar{\mathcal{E}}_a^{(2)}$ below $u(x,t)$ (which intersects $u/v=0$ at $\psi=0$ in Fig. 2). This definition gives distribution functions that, through first order in ϵ, satisfy the Vlasov equation uniformly since they are written in terms of $\bar{\mathcal{E}}_a^{(1)}$ and $\bar{\mathcal{E}}_a^{(2)}$, and have continuous first derivatives even along the matching curve $u(x,t)$, since $\bar{\mathcal{E}}_a^{(1)}$ and $\bar{\mathcal{E}}_a^{(2)}$ meet smoothly there as indicated by the curves near $u(x,t)$ in Fig. 2. This figure also doubles as a plot of the level curves of $f_a^{(+)}$ when evaluated on the Poincaré surface.

If these distribution functions are to describe a self-consistent solution to the Vlasov-Poisson-Ampère equations, they also must yield charge and current densities $\rho(x,t)$ and $j(x,t)$ which generate the correct self-consistent superimposed electric potential φ^L. Thus, neglecting quantities higher than first order in ϵ, Eqs. (2) and (3) must be satisfied by the distribution functions $f_a=f_a^{(+)}$ of Eq. (8) when $E=-\partial\varphi^L/\partial x$. That these conditions for self-consistency are indeed satisfied through first order can be shown by using the detailed definitions of the single-wave BGK functions $g_a^{(i)}$ for small amplitude waves which are discussed, for instance, in Ref. [5]. But this result can be established more easily, since the integrals in Eqs. (2) and (3) can be calculated conveniently but still correctly to first order in ϵ simply by replacing $f_a^{(+)}$ with f_a^L, the errors so introduced being $\sim O(\epsilon^{3/2})$. This is so because the linear theory is adequate outside the regions where particles are trapped by each individual wave, regions where u satisfies $m_a(u-v_i)^2/2\leq 2\epsilon|q_a\bar{\varphi}_i|$ for $i=1,2$, and the functions $f_a^{(+)}$ agree to first order outside these regions with the linearly superimposed distribution functions $f_a^L=F_a(u)+h_a^{(1)}\times(\mathcal{E}_a^{(1)})+h_a^{(2)}(\mathcal{E}_a^{(2)})$. On the other hand, the difference between $f_a^{(+)}$ and f_a^L inside the trapping regions is $\sim O(\epsilon)$. But, from the expression above, the widths in velocity $\Delta u^{(i)}$ of the trapping regions satisfy $\Delta u^{(i)}\sim O(\epsilon^{1/2})$. Thus, since each component wave independently satisfies the Poisson and Ampère equations, which requires that each pair (ω_1,k_1) and (ω_2,k_2) be a root of the Vlasov dispersion relation of Eq. (4), $f_a^{(+)}$ can be replaced by the more convenient f_a^L to establish the self-consistency of the result.

Thus, we have indeed obtained a self-consistent superimposed solution that describes a plasma state containing two small amplitude periodic BGK waves. This solution embodies a nonlinear superposition principle, in which the single-wave potentials are superimposed linearly, $\varphi^L = \varphi^{(1)} + \varphi^{(2)}$, while the distribution functions $f_a^{(+)}$ are constructed from the single-wave distributions by the nonlinear rule given explicitly by Eq. (8). Extension of this development to the nonlinear superposition of N small amplitude waves is straightforward and leads to a simple generalization of Eq. (8) based on $g_a^{(i)}(\overline{\mathscr{E}}_a^{(i)})$, $i=1,2,\ldots,N$, where the approximate invariants $\overline{\mathscr{E}}_a^{(i)}$ are defined on an appropriate set of N nonoverlapping regions that cover the x-u plane.

Returning to the two-wave case, the conditions under which this superposition principle holds can be surmised by more detailed consideration of $\overline{\mathscr{E}}_a^{(1)}$ and $\overline{\mathscr{E}}_a^{(2)}$. Over the regions in which they are used in the definition of Eq. (8), these quantities are approximate invariants, with errors that are at most $\sim O(\epsilon^2/\delta v^2)$. For $\overline{\mathscr{E}}_a^{(1)}$ and $\overline{\mathscr{E}}_a^{(2)}$ to be true first-order invariants thus requires that $\delta v \sim O(\epsilon^s)$ where $s < \frac{1}{2}$. Reverting to the physical wave amplitudes $\overline{\varphi}_1$ and $\overline{\varphi}_2$ (setting $\epsilon = 1$), this condition takes the dimensionless form $\overline{\varphi}_1, \overline{\varphi}_2 \ll (m_a/q_a)\delta v^2$ which must hold for all species in the plasma. Physically this reflects the fact that two waves interact less strongly the larger their relative phase velocity, since large δv means that particles trapped in one wave feel only a high frequency perturbation from the passing field of the other, and thus on average are affected very little.

The superimposed solutions constructed here are relevant to recent experiments investigating the role of two-wave induced dynamical chaos in nonlinear plasma heating [7]. More directly, they also appear to describe certain time-asymptotic plasma states that have been observed in long-time numerical simulations [6] of a large amplitude electrostatic plasma wave undergoing nonlinear Landau damping [8] in a Maxwellian plasma. In these simulations, the electric field is observed to approach a standing wave pattern, while the distribution functions grow vortex structures corresponding to particles which are trapped in each of two waves of equal but opposite velocity. Furthermore, asymptotic states with similar characteristics, although not symmetric, also are observed in the nonlinear saturation of linear instabilities such as in the single-sided bump-on-tail or two-stream distributions [6]. Physical states such as these appear to contain two small amplitude BGK waves and be well described by self-consistent distribution functions of the two-wave type developed here.

In summary, by employing the approximate invariant quantities $\overline{\mathscr{E}}_a^{(i)}$ for the two-wave superimposed field, we have constructed the smooth approximate particle distribution functions, given by Eq. (8), which satisfy the Vlasov equation uniformly through first order in the wave amplitudes, and yield through the Poisson and Ampère equations the corresponding correct linearly superimposed self-consistent field. This result shows that the notion of superimposed small amplitude BGK waves is meaningful, even though the superposition is not linear. These superimposed solutions appear to be essential to the proper description of the time-asymptotic states of some large amplitude plasma waves that undergo nonlinear Landau damping [6].

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