Momentum Analyticity and Finiteness of the 1-Loop Superstring Amplitude

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The type II superstring amplitude to 1-loop order is given by an integral of ϑ functions over the moduli space of tori, which diverges for real momenta. We construct the analytic continuation which renders this amplitude well defined and finite, and we find the expected poles and cuts in the complex momentum plane.

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One of the key successes of superstring theory is the observation early on that the type II superstring amplitude to 1-loop order does not exhibit the tachyon and dilaton divergences that occur in the bosonic string [1]. Nowadays, all superstring amplitudes are believed to be finite in perturbation theory [2,3]. Yet, even the nature of the most basic 1-loop amplitude for the scattering of four massless bosonic strings, such as the graviton, dilaton, and antisymmetric tensor, has so far not been completely elucidated (see, e.g., [4]). This amplitude is represented by an integral over positions of vertex operators and moduli of the torus, which is absolutely convergent only when the Mandelstam variables s, t, and u are all purely imaginary. For real momenta, the integral is real and infinite.

Reality of the 1-loop amplitude would imply the vanishing of the tree-level four-point function by the optical theorem, which is in contradiction with its known nonvanishing expression. Both the reality and the divergence of the integral representation are manifestations of the same illness: The integral representation has not been properly analytically continued in its dependence on external momenta. Another way to say this is that unlike in quantum field theory, one has not properly provided an $i\epsilon$ prescription for the string propagators [5]. This issue was addressed in the field theory limit of superstring theory in [6,7]. Other attempts at resolving this and related problems are in [8].

In this Letter, we present the crucial steps and results in the construction of the analytic continuation of these integral representations. Specifically, we consider the amplitude for the scattering of four external massless bosons, including the graviton, dilaton, and antisymmetric tensor, to 1-loop order in the type II superstring. This is the simplest nonvanishing on-shell amplitude, and the first nontrivial quantum gravity loop amplitude which is both finite and unitary. We obtain explicit formulas for the singularities in the complex momentum plane in the form of poles in s, t, and u and cuts along the positive real axis. The amplitude may be represented by a double dispersion relation and the (double) spectral density is evaluated explicitly. Generalization to N-particle 1-loop amplitudes is technically more involved, but straightforward.

Our starting point is the amplitude $A(k_1,\ldots,k_4)$ for the type II superstring to 1-loop order for four external massless bosons of momenta k_i, $i=1,\ldots,4$. The amplitude has a kinematical prefactor—the same at tree level and irrelevant to our considerations—and a single invariant amplitude, given as an integral over the positions of the vertex operators and the moduli of the world-sheet [1-3]. The world-sheet at 1-loop order is a torus, which can be represented as a parallelogram M_τ in the complex plane with corners at $0,1,\tau,1+\tau$, and opposite sides identified. The parameter $\tau=\tau_1+i\tau_2$ is the complex modulus. Modular invariance restricts τ to the fundamental domain $\{\tau_2>0\}/PSL(2,\mathbb{Z})$, which we choose to be $D=\{\tau_2>0,|\tau|>1,|\tau_1|<\frac{1}{2}\}$. Let $G(z,w)$ be the Green's function on the torus defined by $-\partial_z\partial_{\bar z}G(z,w)=2\pi\delta(z,w)-2\pi/\tau_2$. Then

$$A(k_1,\ldots,k_4)=A(s,t,u)=\int_D\frac{d^2\tau}{\tau_2^2}\int_{M_\tau}\prod_{i=1}^{4}\frac{d^2z_i}{\tau_2}\prod_{i<j}\exp\{\tfrac{1}{2}s_{ij}G(z_i,z_j)\}\,. \tag{1}$$

Here we have introduced standard Mandelstam variables $s=s_{12}=s_{34}=-(k_1+k_2)^2$, $t=s_{23}=s_{14}=-(k_2+k_3)^2$, $u=s_{13}=s_{24}=-(k_1+k_3)^2$, where all external momenta are on mass shell $k_i^2=0$, so that $s+t+u=0$. The Green's function may be expressed in terms of ϑ functions

$$e^{-G(z,w)}=\left|\frac{\vartheta_1(z-w,\tau)}{\vartheta_1'(0,\tau)}\right|^2\exp\{-2\pi[\mathrm{Im}(z-w)]^2/\tau_2\}\,. \tag{2}$$

By translation invariance we may set $z_4=0$. The region of integration separates naturally into six regions defined by the various orderings of $\mathrm{Im}z_1,\mathrm{Im}z_2,\mathrm{Im}z_3$. The amplitude can be split accordingly as [6,7]

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$$A(s,t,u)=2A(s,t)+2A(t,u)+2A(u,s) \tag{3}$$

with $A(s,t)$ given by the integral in (1) restricted to the region $\mathrm{Im}z_1 \le \mathrm{Im}z_2 \le \mathrm{Im}z_3$ and $u=-s-t$. The amplitude $A(s,t)$ converges absolutely for $-2<\mathrm{Re}s, \mathrm{Re}t \le 0$. It suffices then to analytically continue $A(s,t)$ in order to obtain a finite amplitude $A(s,t,u)$ with manifest duality. This is the problem we focus on.

The first step in the analytic continuation is a careful analysis of the singularities of (1) as τ tends to the boundary of moduli space. For this we need the real variables $0 \le x,y \le 1$ defined by $z=x+\tau y$, in terms of which $G(z,w)$ becomes

$$\begin{aligned}\exp[-G(z,0)]&=\frac{1}{4\pi^2}|q|^{y^2-y}|1-e^{2\pi ix}q^y|^2|1-e^{-2\pi ix}q^{1-y}|^2R(z)\,,\\ R(z)&=\prod_{n=1}^{\infty}|1-q^n|^{-4}|1-e^{2\pi ix}q^{n+y}|^2|1-e^{-2\pi ix}q^{n+1-y}|^2.\end{aligned} \tag{4}$$

Here we have set $q\equiv e^{2\pi i\tau}$. Since $|q|<e^{-\pi\sqrt{3}}$ in D, $R(z)$ is a nowhere vanishing, bounded function of both z and q. Thus the expected logarithmic singularity of the Green's function $G(z,w)$ when z approaches w corresponds to the intermediate two factors in (4), while the term $|q|^{y^2-y}$ describes the behavior of the Green's function near the boundary of moduli space. It is now convenient to introduce the variables $u_1=y_1$, $u_2=y_2-y_1$, $u_3=y_3-y_2$, $u_4=1-y_3$ and $\alpha_i=2\pi(x_i+y_i\tau_1)$, $i=1,2,3$, and $\alpha_4=2\pi\tau_1-\alpha_1-\alpha_2-\alpha_3$ and to rewrite the amplitude $A(s,t)$ as

$$A(s,t)=\int_D \frac{d^2\tau}{\tau_2^2}\int_0^1\prod_{i=1}^{3}\frac{d\alpha_i}{2\pi}\int_0^1\prod_{i=1}^{4}du_i\,\delta\left(1-\sum_{i=1}^{4}u_i\right)|q|^{-(su_1u_3+tu_2u_4)}\mathcal{J}(u,\alpha,q)\mathcal{R}(u,\alpha,q)\,, \tag{5}$$

where we have defined the functions

$$\begin{aligned}\mathcal{J}(u,\alpha,q)=&|1-e^{i\alpha_1}|q|^{u_1}|^{-t}|1-e^{i\alpha_2+i\alpha_3+i\alpha_4}|q|^{u_2+u_3+u_4}|^{-t}|1-e^{i\alpha_1+i\alpha_2}|q|^{u_1+u_2}|^{s+t}\\ &\times|1-e^{i\alpha_2}|q|^{u_2}|^{-s}|1-e^{i\alpha_1+i\alpha_3+i\alpha_4}|q|^{u_1+u_3+u_4}|^{-s}|1-e^{i\alpha_1+i\alpha_4}|q|^{u_1+u_4}|^{s+t}\\ &\times|1-e^{i\alpha_3}|q|^{u_3}|^{-t}|1-e^{i\alpha_1+i\alpha_2+i\alpha_4}|q|^{u_1+u_2+u_4}|^{-t}|1-e^{i\alpha_2+i\alpha_3}|q|^{u_2+u_3}|^{s+t}\\ &\times|1-e^{i\alpha_4}|q|^{u_4}|^{-s}|1-e^{i\alpha_1+i\alpha_2+i\alpha_3}|q|^{u_1+u_2+u_3}|^{-s}|1-e^{i\alpha_3+i\alpha_4}|q|^{u_3+u_4}|^{s+t}\,,\\ \mathcal{R}(u,\alpha,q)=&\left(\frac{R(z_2-z_1)R(z_3)}{R(z_3-z_1)R(z_2)}\right)^{-s}\left(\frac{R(z_3-z_2)R(z_1)}{R(z_3-z_1)R(z_2)}\right)^{-t}.\end{aligned} \tag{6}$$

A preliminary key observation is that for fixed τ, the integrals in α_i,u_i can be analytically continued to meromorphic functions of s and t on the whole plane. In fact the condition $|q^{u_1+u_2+u_3+u_4}|=|q|\le e^{-\pi\sqrt{3}}$ guarantees that at most 6 of the 12 factors in $\mathcal{J}(u,\alpha,q)$ can approach 0 simultaneously, so that the following lemma and its variants will do the job.

Lemma 1.—The integral over three complex variables λ_i, $i=1,2,3$, given by

$$\int_{|\lambda_i|<1}d^2\lambda_i|1-\lambda_1|^{-t}|1-\lambda_2|^{-s}|1-\lambda_3|^{-t}|1-\lambda_1\lambda_2|^{s+t}|1-\lambda_2\lambda_3|^{s+t}|1-\lambda_1\lambda_2\lambda_3|^{-s}E(\lambda_i)\,, \tag{7}$$

where $E(\lambda_i)$ is a smooth function, can be analytically continued to the whole plane, with at most poles for s, t, and $u=-s-t$ at half integers greater than or equal to 2. The coefficients of a pole in one variable are entire in the other variable.

We should note here though that these meromorphic terms without cuts are required as part of the amplitude simply in view of the fact that they are the ones that reproduce correctly the large s and t behavior of the 1-loop amplitudes [9]. The remaining contributions with cuts have only power law behavior.

We concentrate now on the analytic continuation of terms requiring cuts, and ignore meromorphic terms which can be treated by Lemma 1. This means that only the region with $\tau_2\to\infty$ is relevant, and we restrict the domain D of integration to the simpler $\{|\tau_1|<\frac{1}{2},\tau_2>1\}$. The next key observation is that it suffices to set $\mathcal{R}(u,\alpha,q)=1$. Indeed it suffices to construct the analytic continuation of $A(s,t)$ to an arbitrary half-space $\mathrm{Re}s<N$, $\mathrm{Re}t<N$. Now for any fixed N, $\mathcal{R}(u,\alpha,q)$ can be expanded as

$$\mathcal{R}(u,\alpha,q)=1+\sum_{\substack{0\le I,J_k,|L_k|\le 2N\\ 1\le \min(I,\max_k J_k)}}C_{I\{J_k\}\{L_k\}}e^{i\sum_{k=1}^{4}L_k\alpha_k}|q|^{I+\sum_{k=1}^{4}J_ku_k}+|q|^{2N+1}E_N(u,\alpha,q)\,,$$

where $E_N(u,\alpha,q)$ is smooth and bounded. In particular the contribution of $|q|^{2N+1}E_N(u,\alpha,q)$ to (5) is convergent and meromorphic for $\mathrm{Re}s<N, \mathrm{Re}t<N$. This reduces $\mathcal{R}$ to the finite sum, all the terms of which can be treated by the method we give below. We work out explicitly the leading term 1, which exhibits branch cuts in s and t starting from 0. The net effect of each term is just to shift the beginning point of each cut to an even positive integer instead of 0. Physi-

cally, the cuts start at the lowest invariant mass^2 at which an intermediate physical string state may be produced.

The leading case corresponds to massless intermediate states, and the perturbative terms to higher mass states. Thus we need only consider the following integral

$$A(s,t)=\int_1^\infty \frac{d\tau_2}{\tau_2^2}\int_0^1 \prod_{i=1}^4 \frac{d\alpha_i}{2\pi}\int_0^\infty \prod_{i=1}^4 du_i\,\delta\left(1-\sum_{i=1}^4 u_i\right)|q|^{-(su_1u_3+tu_2u_4)}\mathcal{J}(u,\alpha,q) \tag{8}$$

up to cuts that start at nonzero positive even integers, which can be treated perturbatively. The integral (8) is symmetric under $(u_1\leftrightarrow u_3,\alpha_1\leftrightarrow\alpha_3)$ and under $(u_2\leftrightarrow u_4, \alpha_2\leftrightarrow\alpha_4)$, so the integration region can be restricted to $u_1\le u_3$ and $u_2\le u_4$ upon including an overall factor of 4. The remaining integral may be split into two regions I and II defined respectively by $(u_1+u_2)\tau_2<1$ and $(u_1+u_2)\tau_2>1$.

In region I, the exponential $|q|^{-(su_1u_3+tu_2u_4)}$ remains a bounded smooth function. In view of Lemma 1, the contribution of each τ_2 (and hence the whole integral, since the $d\tau_2/\tau_2^2$ measure is finite) is a meromorphic function of s and t in the whole plane.

The difficult region is then II, which will generate both poles and cuts. Here, however, the factor $\mathcal{J}(u,\alpha,q)$ simplifies considerably since $|q|^{u_i+u_j}<e^{-2\pi}$ whenever one index i or j is odd and the other even, and we can expand in these variables. For s and t in a fixed arbitrary strip, we need keep only a finite number of terms. Each term can be treated in the same way as the main term, producing cuts in *both* s and t shifted to the right by an even integer. Restricting ourselves to the main term, we need keep only the four factors $(1-e^{i\alpha_i}|q|^{u_i})^{\text{power}}$, $i=1,\ldots,4$ in (6). At this point it is convenient to enlarge the region II back to the full region I+II, since the contribution of I with four factors is again just a meromorphic function in the entire plane. The angular α_i dependence decouples and integrates out to a hypergeometric function. Thus up to meromorphic functions on the whole plane and higher cuts the original amplitude $A(s,t)$ can be expressed as

$$A(s,t)=\int_0^\infty \frac{d\tau_2}{\tau_2^2}\int_0^1 \prod_{i=1}^4 du_i\,\delta\left(1-\sum_{i=1}^4 u_i\right)|q|^{-(su_1u_3+tu_2u_4)}F\left(\frac{s}{2},\frac{s}{2};1,|q|^{2u_2}\right)$$
$$\times F\left(\frac{s}{2},\frac{s}{2};1,|q|^{2u_4}\right)F\left(\frac{t}{2},\frac{t}{2};1,|q|^{2u_1}\right)F\left(\frac{t}{2},\frac{t}{2};1,|q|^{2u_3}\right). \tag{9}$$

We present the analytic continuation of (9) under two forms, each with its own advantages. In the first, the cuts along the positive axis can be written out explicitly in terms of logarithms. An important ingredient is the Mellin transform of hypergeometric functions, which we define for each fixed integer N by

$$f_N(s,\alpha)=\int_0^1 dx\,x^{-1-\alpha}\left[F\left(\frac{s}{2},\frac{s}{2};1,x\right)-\sum_{k=0}^{N-1}C_k(s)x^k\right],\quad C_k(s)=\frac{\Gamma(s/2+k)^2}{\Gamma(s/2)^2\Gamma(k+1)^2}. \tag{10}$$

The polynomials $C_k(s)$ are the expansion coefficients of F and arise naturally as the residue of the tree-level closed superstring amplitude at $t=2k$.

Lemma 2.—The function $\partial_\alpha^n f_N(s,\alpha)$ is the sum of a meromorphic function $f_{N,n}^+(s,\alpha)$ of α with poles of order $n+1$ at $N,N+1,\ldots$, and a meromorphic function $f_{N,n}^-(s,\alpha)$ of s with simple poles at the even integers greater than or equal to $n+2,n+4,\ldots$. The dependence on the other variable in each case is entire.

The analytic continuation of $A(s,0)$ to $\mathbb{C}\backslash\mathbb{R}_+$ can now be expressed simply as

$$A(s,0)=\frac{\pi}{6}s^3\ln(-s)\int_0^1 du(1-u)^3u^3f_0(s,-\tfrac{1}{2}su)^2 \tag{11}$$

up to a meromorphic function and higher cuts. We note in particular that $A(s,0)$ has double poles at $s=2,4,\ldots$ lying under the cuts and that its residues depend on the cut discontinuities. In particular the mass renormalizations and widths of massive string states arise in this way. To obtain the analytic continuation in both s and t, we begin by introducing the following expression in four variables, s, s', t, and t',

$$B_N(s,s';t,t')=-2\pi\sum_{k_1,k_2=0}^\infty C_{k_1}(t)C_{k_2}(s)$$
$$\times\int\!\!\int_{2k_iu_i<N,u_1+u_2<1} du_1du_2\left[\sum_{k_3,k_4}^{2N}C_{k_3}(t)C_{k_4}(s)\frac{\Phi(-s'u_1+2k_3)-\Phi(-t'u_2+2k_4)}{s'u_1-t'u_2-2k_3+2k_4}\right.$$
$$+\sum_{k=0}^{N/2-1}C_k(s)\Phi(-t'u_2+2k)f_{2N}[t,\tfrac{1}{2}(-s'u_1+t'u_2-2k)]$$
$$\left.+\sum_{k=0}^{N/2-1}C_k(t)\Phi(-s'u_1+2k)f_{2N}[s,\tfrac{1}{2}(s'u_1-t'u_2-2k)]\right], \tag{12}$$

where the function Φ is defined by

$$\Phi(\zeta)=[\zeta(1-u_1-u_2)+2k_1u_1+2k_2u_2]^2\ln[\zeta(1-u_1-u_2)+2k_1u_1+2k_2u_2]\,.$$

It is readily seen that $\partial_{s'}^{n}\partial_{t'}^{n}B_N(s,s';t,t')$ is holomorphic in the region $\mathrm{Re}\,s,\mathrm{Re}\,t<2+n, s',t'\in\mathbb{C}\backslash\mathbb{R}_+$, $\mathrm{Re}\,s',\mathrm{Re}\,t'<N$, and N is large enough compared to n. In fact $B_2(s,s;t,t)$ gives precisely the desired continuation of $A(s,t)$ to the half-plane $\mathrm{Re}\,s,\mathrm{Re}\,t<2$ up to a meromorphic function. The analytic continuation of $A(s,t)$ to an arbitrary strip can be obtained by integrating (12) with respect to s' and t'; e.g., for the strip $|\mathrm{Re}\,s|,|\mathrm{Re}\,t|<3$ it can be expressed as

$$A(s,t)=A(s,0)+A(0,t)+\int_0^t dt'\int_0^s ds'\,\partial_{s'}\partial_{t'}B_3(s,s';t,t')+C_1(s,t)\,. \tag{13}$$

The additional term $C_1(s,t)$ consists of functions which are either meromorphic or have only logarithmic cuts starting from $s=2$ or $t=2$, and can be worked out explicitly. Thus (11) and (13) give the desired exact formulas for cuts starting at 0. Cuts corresponding to higher intermediate mass states can be obtained successively by the same method.

The second method of obtaining the analytic continuation is to recast expression (9) in the form of a double dispersion relation

$$A(s,t)=\int_0^\infty d\sigma\int_0^\infty d\tau\,\frac{\rho_{s,t}(\sigma,\tau)}{(s-\sigma)(t-\tau)}\,, \tag{14}$$

where we have again ignored meromorphic terms. The (double) spectral density $\rho_{s,t}(\sigma,\tau)$ is given by

$$\rho_{s,t}(\sigma,\tau)=\int_0^\infty d\beta_1\int_0^\infty d\beta_2\,\varphi(t,\beta_1)\varphi(s,\beta_2)\int_0^1 du_1\int_0^{1-u_1}du_2(1-u_1-u_2)^2 \times\int_{x_0}^\infty dx\,(x-x_0)^2\theta(u_1\sigma-2x) \times\theta(u_2\tau-2x)\varphi\left(t,\frac{u_1\sigma}{2}-x\right)\varphi\left(s,\frac{u_2\tau}{2}-x\right) \tag{15}$$

with $x_0\equiv(u_1\beta_1+u_2\beta_2)(1-u_1-u_2)^{-1}$, and $\varphi(s,\beta)$ is the inverse Laplace transform of the hypergeometric function $F(s/2,s/2;1,x)$. Since $\varphi(s,\beta)=f_0(s,\beta+i\epsilon)-f_0(s,\beta-i\epsilon)$, the spectral density is a meromorphic of s and t with poles at even integers. As $\sigma,\tau\to\infty$, $\rho_{s,t}(\sigma,\tau)$ grows linearly, so that the integral (15) is not convergent. We can, however, obtain convergence by subtracting a suitable meromorphic function, a procedure analogous to (13). The net outcome is that the double dispersion integral defines an obvious analytic continuation to the full s,t complex plane with a cut on the real positive s and t axes, with known discontinuity $\rho_{s,t}(\sigma,\tau)$. As a simple example we return to the case of $t=0$. Up to a meromorphic function the double dispersion reduces to a simple dispersion relation with spectral density

$$\rho_s(\sigma)=-\int_0^\infty\frac{d\tau}{\tau}\rho_{s,0}(\sigma,\tau) =\frac{\sigma^3}{24}\int_0^1 du\,u^3(1-u)^3[f_0(s,-\tfrac{1}{2}u\sigma)]^2\,, \tag{16}$$

which gives back the singularities obtained earlier in (11). In view of the decomposition of f_0 into $f_0^\pm$ of Lemma 2, these singularities have a natural interpretation in terms of underlying Feynman diagrams. The contribution of $(f_0^+)^2$ contains double poles in s and corresponds to the vacuum polarization diagrams in the s channel. This is the only diagram that needs subtraction. The contribution $2f_0^+f_0^-$ corresponds to the insertion of the triangle diagram with a single simple pole left, and finally $(f_0^-)^2$ corresponds to the box diagram, and no poles in s occur. In summary:

Theorem.—The amplitude $A(s,t)$ can be analytically continued to the region $(s,t)\in(\mathbb{C}\backslash\mathbb{R}_+)^2$. In this region it has poles when $s+t$ is an integer less than or equal to -2. The jumps across the cut along the positive real axis together with the underlying poles at even positive integers can be read off either in terms of logarithms as in (13), or in terms of a dispersion relation as in (14).

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