Critical Superstring Vacua from Noncritical Manifolds: A Novel Framework for String Compactification and Mirror Symmetry

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A new framework is found for the compactification of supersymmetric string theory. It is shown that the massless spectra of critical string vacua with central charge $c=3D_{\rm crit}$ can be derived from manifolds of complex dimension $D_{\rm crit}+2(Q-1)$, $Q\geq 1$, whose first Chern class is quantized in a particular way. This new class is more general than that of Calabi-Yau manifolds because it contains spaces corresponding to vacua with no Kähler deformations, i.e., no antigenerations, thus providing mirrors of rigid Calabi-Yau manifolds. The constructions introduced here lead to new insights into the relation between Landau-Ginzburg vacua on the one hand and Calabi-Yau manifolds on the other.

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It is believed that the heterotic string without torsion can propagate consistently in a manifold only if this manifold is complex, Kählerian, and admits a covariantly constant spinor, i.e., has vanishing first Chern mass. Manifolds of this type, so-called Calabi-Yau manifolds, are examples of left-right symmetric string vacua with $N=2$ supersymmetry on the world sheet. It is further believed that the configuration space of such ground states features an important symmetry, not at all manifest in the construction of the superstring: mirror symmetry. The predictions derived from this symmetry, which has been discovered in the context of Landau-Ginzburg (LG) vacua in [1] and proven to exist in this framework in [2], have been shown to be correct in all computations performed so far [3,4]. Independent evidence for this symmetry has been found in the context of orbifolds of exactly solvable tensor models [5].

Mirror symmetry creates a puzzle. There are well-known Calabi-Yau vacua which are rigid, i.e., they do not have string modes corresponding to complex deformations of the manifold, fields that describe generations in the low energy theory. Since mirror symmetry exchanges complex deformations and Kähler deformations of a manifold, the latter describing the antigenerations seen by a four-dimensional observer, it would seem that the mirror of a rigid Calabi-Yau manifold cannot be Kählerian and hence does not describe a consistent string vacuum. It follows that the class of Calabi-Yau manifolds is not the appropriate setting in which to discuss mirror symmetry and the question arises what the proper framework might be.

It is the purpose of this Letter to introduce a new class of manifolds which generalizes the class of Calabi-Yau spaces of complex dimension $D_{\rm crit}$ in a natural way. The manifolds involved are of complex dimension $[D_{\rm crit}+2(Q-1)]$, where Q is a natural integer, and have a positive first Chern class which is quantized in multiples of the degree of the manifold. Thus they do not describe, *a priori*, consistent string ground states. Surprisingly, however, it is possible to derive from these higher dimensional manifolds the massless modes of critical string vacua. This can be done not only for the generations but also for the antigenerations. For particular types of these new manifolds it is in fact possible to construct $D_{\rm crit}$-dimensional Calabi-Yau manifolds directly from the $[D_{\rm crit}+2(Q-1)]$-dimensional spaces.

This new class of manifolds is, however, not in one-to-one correspondence with the class of Calabi-Yau manifolds as it also contains manifolds which describe string vacua that do not contain massless modes corresponding to antigenerations. It is precisely this new type of manifold that is needed in order to construct mirrors of rigid Calabi-Yau manifolds without generations. The results presented in this Letter suggest that the noncritical manifolds described here are no less physical than critical manifolds and indeed define the appropriate generalization of the Calabi-Yau framework of string compactification. They also lead to important ramifications regarding the relation between Landau-Ginzburg theories and critical manifolds.

Higher dimensional manifolds with quantized positive first Chern class.—Consider the class of manifolds $\mathcal{M}_{N,d}$ of complex dimension N embedded in a weighted projective space $\mathbb{P}_{(k_1,\ldots,k_{N+2})}$ as hypersurfaces defined by the zero locus of a transverse polynomial $p(z_1,\ldots,z_{N+2})$ of degree d. Here the integers k_i describe the weights of the ambient weighted projective space. It will be assumed that the weights k_i and the degree d are related via the constraint

$$\sum_{i=1}^{N+2} k_i = Qd\,, \tag{1}$$

where Q is a positive integer. Relation (1) is the defining property of the class of manifolds I will consider in this Letter. It is a rather restrictive condition in that it excludes many types of varieties which are transverse and even smooth but are not of physical relevance. (A particularly simple manifold in this class, the cubic sevenfold $\mathbb{P}_8$ [3], has been the subject of recent investigations [6–8].)

Alternatively, manifolds of the type above may be characterized via a curvature constraint. Because of (1) the first Chern class is given by $c_1(\mathcal{M}_{N,d})=(Q-1)dh$, where h is the pullback of the Kähler form of the ambient space. Hence the first Chern class is quantized in multiples of the degree of the hypersurface. For $Q=1$ the first Chern class vanishes and the manifolds for which condition (1) holds are Calabi-Yau manifolds, defining consistent ground states of the supersymmetric closed string. For $Q>1$ the first Chern class is nonvanishing and therefore these manifolds cannot possibly describe vacua of the critical string, or so it seems.

It will be shown below that these spaces are closely related to string vacua of critical dimension $D_{\rm crit}=N-2\times(Q-1)$. The evidence for this is twofold. First it is possible to derive from these higher dimensional manifolds the massless spectrum of critical vacua. Furthermore it is shown that it is possible to construct Calabi-Yau manifolds $M_{\rm CY}$ of dimension $D_{\rm crit}$ and complex codimension Q directly from certain subclasses of hypersurfaces of type (1). In terms of the critical dimension and the codimension the class of manifolds to be investigated below can be described as the projective configurations

$$\mathbb{P}_{(k_1,\ldots,k_{(D_{\rm crit}+2Q)})}\left[\frac{1}{Q}\sum_{i=1}^{D_{\rm crit}+2Q}k_i\right],\qquad(2)$$

where the expression in brackets denotes the degree of the polynomial.

The class defined by (2) contains manifolds with no antigenerations. Hence it is necessary to have some way other than Calabi-Yau manifolds to represent string ground states in order to establish a relation between such higher dimensional manifolds and string vacua. One possible way to achieve this is to associate them to Landau-Ginzburg theories which are characterized by their chiral ring structure encoded in the superpotential.

In certain benign situations the subring of monomials of charge 1 in the chiral ring describes the generations of the vacuum [9]. Thus the generations are easily derived for this subclass of theories in (2) because the polynomial ring is identical to the chiral ring of the corresponding LG theory. In general a more sophisticated analysis, involving the singularity structure of the higher dimensional spaces, will have to be done [10].

It remains to extract the second cohomology. For manifolds with positive first Chern class all of the second cohomology resides in $H^{(1,1)}$. The simplest example in the class (2), $\mathbb{P}_8[3]$, already shows, however, that there is a mismatch between the Kähler sector of the higher dimensional manifolds and that of the critical vacuum: For $\mathbb{P}_8[3]$ the diagonal cohomology leads to $h^{(p,p)}=1, 0\le p\le 7$, whereas the critical vacuum, described by the tensor model 1^9, has no Kähler deformations at all. Since the critical theory does not contain modes corresponding to (1,1) forms it appears that a potential manifold cannot be Kählerian and hence not projective. Thus it seems that the seven-dimensional manifold $\mathbb{P}_8[3]$, whose polynomial ring is identical to the chiral ring of the LG theory, is useful merely as an auxiliary device in order to describe one sector of the critical LG string vacuum.

Relation between critical and noncritical manifolds.—It turns out that by looking at the manifolds of the type described by (2) in a particular way it is indeed possible to extract the second cohomology in a canonical manner (even if there is *none*). The way this works is as follows: The manifolds (2) will, in general, not be described by smooth spaces but will have singularities which arise from the projective identification. The basic idea is that the *critical* string physics has its origin in the singularity structure and the chiral ring of these higher dimensional *noncritical* manifolds. In particular the antigenerations are generated by the singularities.

In the following I will make the ideas involved more precise and illustrate how they work with a few particularly simple classes of theories, leaving a more detailed investigation of other types of manifolds to a more extensive discussion [10]. As an unexpected bonus this derivation will provide new insight into the Landau-Ginzburg/Calabi-Yau connection.

It is useful to first consider an example in some detail. The GSO projected LG theory based on the superpotential

$$W=\sum_{i=1}^{3}(x_i^3y_i+y_i^3)+y_4^3\qquad(3)$$

describes a vacuum with 35 generations and 8 antigenerations. Associated to this ground state is the affine configuration $\mathbb{C}_{(2,2,2,3,3,3,3)}$ [9] which induces, via projectivization, a five-dimensional weighted hypersurface $\mathbb{P}_{(2,2,2,3,3,3,3)}$ [9]. This compact manifold has orbifold singularities with respect to the cyclic groups $\mathbb{Z}_3$ and $\mathbb{Z}_2$, the first one obtained by setting $x_i=0$, the latter by setting $y_i=0$:

$$\mathbb{Z}_3\colon \mathbb{P}_3[3]\ni\left\{p_1=\sum_{i=1}^{4}x_i^3=0\right\};\quad \mathbb{Z}_2\colon \mathbb{P}_2.\qquad(4)$$

The $\mathbb{Z}_3$-singular set is a smooth cubic surface which supports seven (1,1) forms whereas the $\mathbb{Z}_2$-singular set is just the projective plane and therefore adds one further (1,1) form. Hence the singularities induced on the hypersurface by the singularities of the ambient weighted projective space give rise to a total of eight (1,1) forms. A simple count leads to the result that the subring of monomials of charge 1 is of dimension 35. Thus we have succeeded in deriving the massless spectrum of the critical theory from the noncritical manifold $\mathbb{P}_{(2,2,2,3,3,3,3)}$ [9].

It is presumably possible to derive this result via a surgery process on the singular space, but more important is, at this point, that the idea introduced above of relating the spectrum of the string vacuum to the singularity structure of the noncritical manifold also makes it possible to derive from these higher dimensional manifolds the Calabi-Yau manifold of critical dimension. This leads to a canonical prescription which allows one to pass from the LG theory to its geometrical counterpart when the model has antigenerations.

Recall that the structure of the singularities of the weighted hypersurface just involves part of the superpotential, namely, the cubic polynomial p_1 which determines the $\mathbb{Z}_3$-singular set described by a surface. The superpotential thus splits naturally into two parts $p=p_1+p_2$, where p_2 is the remaining part of the polynomial. The idea is to consider the product $\mathbb{P}_3\,[3]\times\mathbb{P}_2$, where the factors are determined by the singular sets of the higher dimensional space, and to impose on this four-dimensional space a constraint described by the remaining part of the polynomial which did not take part in constraining the singularities of the ambient space. In the case at hand this leaves a polynomial of bidegree (3,1) and hence we are led to a manifold embedded in

$$\begin{matrix}\mathbb{P}_2\\ \mathbb{P}_3\end{matrix}\begin{bmatrix}3 & 0\\ 1 & 3\end{bmatrix}\ni\begin{Bmatrix}p_1=y_1^3x_1+y_2^3x_2+y_3^3x_3=0\\ p_2=\sum_{i=1}^4 x_i^3=0\end{Bmatrix}. \tag{5}$$

But this is a well-known Calabi-Yau manifold of complex dimension 3, introduced in [11].

The ideas just described are general. A class of manifolds of a different type which can be discussed in this framework rather naturally is defined by the projective configurations

$$\mathbb{P}_{(2k,K-k,2k,K-k,2k_3,2k_4,2k_5)}[2K]\ni\left\{\sum_{i=1}^{2}(x_i^{K/k}+x_iy_i^2)+x_3^{K/k_3}+x_4^{K/k_4}+x_5^{K/k_5}=0\right\}, \tag{6}$$

where $K=k+k_3+k_4+k_5$ and it is assumed, for simplicity, that K/k and K/k_i are integers. (These conditions can be removed as will be shown in [10].) The singularities in these manifolds are of two types,

$$\mathbb{Z}_2\colon \mathbb{P}_{(k,k,k_3,k_4,k_5)}[K]\ni\left\{p_1=\sum_{i=1}^{5}x_i^{K/k_i}=0\right\};\quad \mathbb{Z}_{K-k}\colon\mathbb{P}_1\,. \tag{7}$$

The $\mathbb{Z}_2$-singular set is a threefold with positive first Chern class embedded in weighted $\mathbb{P}_4$ whereas the $\mathbb{Z}_{K-k}$-singular set is just the sphere $S^2\sim\mathbb{P}_1$.

In complete analogy with the previous discussion the manifolds in this class lead to critical manifolds embedded in

$$\begin{matrix}\mathbb{P}_1\\ \mathbb{P}_{(k,k,k_3,k_4,k_5)}\end{matrix}\begin{bmatrix}2 & 0\\ k & K\end{bmatrix}\ni\begin{Bmatrix}p_1=y_1^2x_1+y_2^2x_2=0\\ p_2=x_1^{K/k}+x_2^{K/k}+x_3^{K/k_3}+x_4^{K/k_4}+x_5^{K/k_5}=0\end{Bmatrix}. \tag{8}$$

That this correspondence is in fact correct can be inferred from the work of [12] where it was shown that these codimension-2 weighted complete intersection manifolds correspond to $N=2$ minimal exactly solvable tensor models.

The picture that emerges from these constructions is the following: Embedded in the higher dimensional manifold is a submanifold which is fibered, the base and the fibers being determined by the singular sets of the ambient manifold. The Calabi-Yau manifold itself is a hypersurface embedded in this fibered submanifold. In more complicated manifolds the singularity structure will consist of hypersurfaces whose fibers and/or bases themselves are fibered, leading to an iterative procedure. The submanifold to be considered will, in those cases, be of codimension larger than 1 and the Calabi-Yau manifold will be described by a submanifold with codimension larger than 1 as well. In the most general situation the fiber bundle will presumably not be a simple product bundle as in the previous examples but will involve nontrivial twists.

The relation between noncritical manifolds of type (2) and critical string vacua is not one to one. By using the construction of "splitting" and "contracting" Calabi-Yau manifolds introduced in [13], it can be shown that noncritical manifolds of different dimensions can lead to one and the same critical vacuum. Thus there exist nontrivial relations between spaces of type (2). A more detailed discussion of these aspects will appear in [10].

In the present framework it becomes clear what is special about string vacua that do not have modes corresponding to antigenerations. Consider again the example related to the tensor model 1^9. Its LG theory describes an affine cubic surface in $\mathbb{C}_9$ the naive compactification of which leads to $\mathbb{P}_8[3]\ni\{\sum_{i=1}^9 z_i^3=0\}$. Counting monomials leads to the spectrum of 84 generations found previously for the corresponding string vacuum and because this manifold is smooth no antigenerations are expected in this model. Hence there does not exist a Calabi-Yau manifold that describes this ground state. A second theory in the space of all LG vacua with no antigenerations is $(2^6)_{A_8^6}^{(0,90)}$ described by a quartic polynomial in $\mathbb{C}_{(1,1,1,1,1,1,2)}$ [4] corresponding again to a smooth manifold with an obviously smooth manifold $\mathbb{P}_{(1,1,1,1,1,1,2)}[4]\ni\{\sum_{i=1}^6 z_i^4+z_7^2=0\}$.

Even though the examples discussed so far are all concerned with critical vacua of central charge $c=9$ and the way they are related to the new class of noncritical spaces of dimension $[3+2(Q-1)]$, it should be clear that these considerations are not specific to this particular set of string ground states. In [10] infinite classes of $(n+1)$-dimensional manifolds will be presented which describe critical vacua of central charge $c=3(n-1)$ and indeed can be shown, via the constructions introduced above, to lead to critical manifolds of complex dimension $(n-1)$.

Mirror symmetry cannot be understood in the framework of Calabi-Yau manifolds. Assuming that this symmetry is indeed a symmetry of the space of left-right symmetric vacua and that the geometrical framework is general enough would lead one to suspect the existence of a space of a new type of noncritical manifolds which contain information about critical vacua, such as the mirrors of rigid Calabi-Yau manifolds. Mirrors of spaces with both sectors, antigenerations and generations are, however, again of Calabi-Yau type and hence the noncritical manifolds which correspond to such ground states should make contact with Calabi-Yau manifolds in some manner.

It has been shown above that the class of higher dimensional Kähler manifolds of type (2) generalizes the framework of Calabi-Yau vacua in the desired way: For particular types of such noncritical spaces Calabi-Yau manifolds of critical dimension are embedded algebraically in a fibered submanifold. For string vacua which cannot be described by Kähler manifolds, and which are mirror candidates of rigid Calabi-Yau spaces, the higher dimensional manifolds still lead to the spectrum of the critical vacuum and a rationale emerges that explains why a Calabi-Yau representation is not possible in such theories. Thus these manifolds of dimension $c/3+2(Q-1)$ define an appropriate framework in which to discuss mirror symmetry.

There are a number of important consequences that follow from the results of the previous sections. First it should be realized that the relevance of noncritical manifolds suggests the generalization of a conjecture regarding the relation between superconformal field theories with $N=1$ spacetime supersymmetry and central charge $c=3D$ on the one hand, and Kähler manifolds of complex dimension D with vanishing first Chern class on the other. It was suggested by Gepner [14] that this relation is one to one. It follows from the results above that instead superconformal theories of the above type are in correspondence with Kähler manifolds of dimension $c/3+2(Q-1)$ with a first Chern class quantized in multiples of the degree.

A second consequence is that the ideas of the section regarding the relation between critical and noncritical manifolds, lead, for a large class of LG theories, to a new canonical prescription for the construction of the critical manifold, if it exists, directly from the 2D field theory.

Recently Batyrev [15] introduced a new construction of mirrors of Calabi-Yau manifolds based on dual polyhedra. This method applies only to manifolds defined by one polynomial in a weighted projective space or products thereof. The method of toric geometry that is used in [15] is, however, not restricted to Calabi-Yau manifolds and therefore the constructions described in the critical/noncritical manifolds section lead to the possibility of extending Batyrev's results to Calabi-Yau manifolds of codimension larger than 1 by proceeding via noncritical manifolds.

It is clear that the emergence in string theory of manifolds with quantized first Chern class should be understood better. The results presented here show that these manifolds are not just auxiliary devices but may be as physical as Calabi-Yau manifolds of critical dimension. In order to probe the structure of these models in more depth it is important to obtain further insight into the complete spectrum of these theories and to compute the Yukawa couplings of the fields. The spectra of the higher dimensional manifolds contain additional modes beyond those that are related to the generations and antigenerations of the critical vacuum, and the question arises what physical interpretation these fields afford.

A better grasp on the complete spectrum of these spaces should also give insight into a different, if not completely independent, approach toward a deeper understanding of these higher dimensional manifolds, which is to attempt the construction of consistent σ models defined via these spaces.

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