Dynamic Model of Onset and Propagation of Fracture

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A two-dimensional model of steady-state crack propagation, in which the stress acting on the fracture surface includes a dissipative term, exhibits a dissipation-dependent effective threshold for fracture. The crack creeps very slowly at external stresses just above the Griffith threshold, and makes an abrupt transition to propagation at roughly the Rayleigh wave speed at higher stresses. When heating due to dissipation is taken into account, the model may exhibit a maximum in the crack propagation speed as a function of applied stress.

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At least two fundamental issues remain unresolved in the modern theory of fracture dynamics [1]. One of these concerns the nature of the Griffith threshold for the onset of crack propagation; a second has to do with propagation speeds and the mechanisms by which they may be limited.

In principle, the Griffith threshold is the externally applied stress at which the flow of elastic energy to the tip of a crack can do just the minimum work necessary for the creation of new fracture surfaces. Measurements of this threshold, however, often yield fracture energies that are appreciably larger than estimates of bare surface energies. This discrepancy usually is ascribed to the extra work done by dissipative forces; but dissipative effects are intrinsically velocity dependent and therefore should play no role in setting a zero-velocity threshold.

A common assumption among experts in fracture mechanics is that, once a crack starts moving in an ideal, defect-free solid, its speed can be limited only by the rate at which stored elastic energy is transported to the crack tip. Thus the limiting speed of a crack must be a sound velocity or, more specifically, the velocity of Rayleigh waves moving along the free fracture surface. Despite its fundamental importance, this assumption has not been tested extensively by experiment. Recent measurements by Fineberg *et al.* [2] indicate that, in at least one plastic material, the limiting fracture speed is significantly less than the Rayleigh velocity, and the approach to this limiting speed is accompanied by the onset of a dynamic instability.

Some insight regarding both of these issues can be obtained from the study of a simple but nontrivial model of steady-crack motion. Consider a two-dimensional elastic material in the (x,y) plane, and suppose that a mode III (antiplane) crack moves along the x axis. The displacement of the material, $u(x,y,t)$, obeys a scalar, massive wave equation of the form

$$\ddot{u} = c^2\nabla^2 u - \omega^2(u-\Delta)\,. \tag{1}$$

Here, c is the wave speed, ω is the "mass," and $\omega^2\Delta$ is an applied force. We include the mass in (1) as a device that allows us to consider a finite applied strain without having to deal explicitly with the outer boundaries of the system. In effect, our material is tied elastically to a substrate or, within a reasonable approximation, the crack is moving along the center line of a strip of finite width. The presence of a small but nonzero ω in (1) implies the existence of a large length scale, say, $W=c/\omega$. For example, W might be the width of the strip or the distance between the material and the substrate.

By definition, $u=0$ along the unbroken portion of the x axis. Far from this axis, or well behind the crack tip where the cohesive forces vanish and the stress is fully relaxed, the displacement u relaxes to Δ. Thus, the externally applied strain at infinity, which is the driving force for crack motion, is $\varepsilon_\infty = \Delta/W$.

To complete the definition of the model, we must specify the traction applied to the fracture surface. The crucial assumption is that this traction can be written in the form

$$\mu \frac{\partial u}{\partial y}\bigg|_{y=0} = \sigma_c\{u\} - \eta \frac{\partial^2 \dot{u}}{\partial x^2}\bigg|_{y=0}\,. \tag{2}$$

The quantity μ on the left-hand side of (2) is an elastic modulus. On the right-hand side, $\sigma_c\{u\}$ is the cohesive stress acting between the open crack faces which, for simplicity, we take to be

$$\sigma_c\{u\} = \begin{cases} \sigma_y, & 0 < u \le \delta, \\ 0, & u > \delta. \end{cases} \tag{3}$$

where u is the crack-opening displacement $u(x,0,t), \sigma_y$ is the yield stress, δ is the range of the cohesive force, and the nominal fracture energy per unit area of fracture surface is $\sigma_y\delta$.

The second term on the right-hand side of (2) is a viscous damping stress acting on the fracture surface. A viscosity of this form is the simplest way of modeling a general dissipative mechanism that depends upon the motion of the system. The two spatial derivatives preserve reflection and translation symmetries, and the single time derivative breaks time-reversal symmetry to produce energy dissipation. Phenomenologically, there is no reason why such a term should not be present in these

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equations of motion; thus, for complex systems of the kind we are considering, we should expect it to occur. There are many more complicated ways of introducing dissipation near the crack tip. Realistic dissipative terms are likely to be nonlinear or, at the very least, η would be some strain-rate-dependent, nonlocal operator. But simplicity is a great asset for the present purposes.

The velocity v for steady-state crack propagation in this model can be obtained by means of Wiener-Hopf methods that are similar to but more elaborate than those described in two earlier papers [3,4]. The details of this calculation, which are interesting in themselves, will be presented in another publication [5].

By far the most interesting result to emerge from this analysis is an approximate relationship between v and the externally applied force:

$$\frac{v/c}{[1-(v/c)^2]^{3/2}} \approx \left(\frac{K_\infty}{K_{\text{eff}}}\right)^{12}. \tag{4}$$

Here, $K_\infty = \varepsilon_\infty \sqrt{W}$ is, apart from a numerical constant and a factor μ, the stress-intensity factor associated with the external force. Technically, K_∞ is a *strain-intensity* factor, which is convenient for our purposes because it contains only geometric information and no constitutive parameters that might depend on temperature or other aspects of the state of the system. Apart from the sound speed c on the left-hand side, all of the system-dependent information in (4) is contained in the quantity K_{eff}:

$$K_{\text{eff}} = (6\delta)^{1/3}\left(\frac{c\eta}{\mu}\right)^{1/12}\left(\frac{\sigma_y}{\mu}\right)^{2/3}. \tag{5}$$

One advantage of writing (4) in terms of the K's is that dependence—or lack thereof—on the macroscopic length W becomes immediately apparent. For example, (4) is valid for applied stresses K_∞ in the range

$$1 \ll \left(\frac{K_\infty}{K_G}\right) \ll \left(\frac{W}{\delta}\right)^{1/6}\left(\frac{\sigma_y}{\mu}\right)^{1/6}, \tag{6}$$

where $K_G = (2\sigma_y\delta/\mu)^{1/2}$ is the stress-intensity factor at the Griffith threshold. Thus we see that the upper bound for validity of (4) is very large for a macroscopic system in which W may be many orders of magnitude larger than any other length scale. On the other hand, K_G and K_{eff} are independent of W and are determined only by constitutive parameters of the material.

The most remarkable aspect of (4) is the large exponent on the right-hand side. If $K_{\text{eff}} > K_G$, so that $K_\infty \cong K_{\text{eff}}$ is within the range of validity (6), then v jumps abruptly from very small values to values near the wave speed c as K_∞ passes through K_{eff}. That is, K_{eff} plays the role of an effective Griffith threshold at which the crack makes what would look experimentally like a sharp transition from slow creep to rapid propagation. The crack in this model does indeed start moving at $K_\infty = K_G$, but this motion may be so slow that an observer would conclude that the threshold is at K_{eff}.

There is a wide range of possibilities for the numerical value of the ratio $K_{\text{eff}}/K_G = (9\sigma_y/2\mu)^{1/6}(c\eta/\mu\delta^2)^{1/12}$. In general, δ is a microscopic length of order angstroms. For a highly ductile material, the viscous length scale $(c\eta/\mu)^{1/2}$ and the yield strain σ_y/μ could be large enough to make this ratio appreciably larger than unity. On the other hand, for a brittle material with a small yield strain and small viscous dissipation, this ratio would be small and the crack would move at almost the wave speed as soon as K_∞ exceeds K_G.

When looked at from a slightly different point of view, this model provides a mechanism by which a small crack inside a material under constant applied strain ε_∞ may remain almost stationary for a long time before nucleating a propagating rupture. In such a situation—a microcrack far from the boundaries of a large system—the relevant stress-intensity factor is proportional to the square root of the crack length L rather than the macroscopic length W. Thus, in a crude but qualitatively reasonable approximation, we may simply replace K_∞ by $\varepsilon_\infty\sqrt{L}$ in (4), and then interpret (4) as a nonlinear equation for L:

$$\frac{\dot{L}/c}{[1-(\dot{L}/c)^2]^{3/2}} \approx \left(\frac{L}{L_{\text{eff}}}\right)^6, \tag{7}$$

where $L_{\text{eff}} \sim (\sigma_y^4\delta^2/\mu^4\varepsilon_\infty^6)^{1/3}(c\eta/\mu)^{1/6}$ is the critical crack length at which runaway fracture begins. This crack grows very slowly for initial values of L that are large enough for the stresses at the crack tips to exceed the Griffith threshold but are appreciably smaller than L_{eff}. As the crack grows, however, the stresses at the tips also grow, and eventually the crack reaches the effective threshold at $L = L_{\text{eff}}$ where $\dot{L}$ quickly accelerates to values of order c. This is the kind of behavior that ordinarily is associated with slow chemical or microstructural changes in the properties of materials under stress. It may be interesting to consider whether some such effects might be purely dynamic in origin.

The transition from slow creep to rapid crack propagation in this model need not be driven only by changes in the applied stress. For example, the yield strain σ_y/μ that appears in K_{eff} is generally a temperature-dependent quantity; thus it seems easy to imagine situations in which shifts between slow and fast fracture are caused by changes in temperature. This would not be the conventional picture of a brittle-to-ductile transition in which there occurs an abrupt change in some constitutive relation. Rather, the constitutive parameters may vary smoothly here, and the abrupt changes may occur in the dynamic response of the system.

An especially interesting possibility is that the heat generated by the viscous dissipation at the crack tip is itself responsible for controlling whether the crack is creeping or propagating rapidly. To explore this possibility we need two additional results from the Wiener-Hopf analysis. First, the crack-opening displacement near the tip is

$$u(x) \approx \frac{\sigma_y x^3}{6\eta v}, \tag{8}$$

and, second, the length of the cohesive zone, that is, the size of the region within which σ_c is nonzero in (3), is

$$l \approx \frac{K_\infty}{\sigma_y}(\mu^3 \eta v)^{1/4}. \tag{9}$$

These results are accurate within the range of applied stresses defined by (6) and, in those circumstances, (8) is accurate out to values of x of order l. For simplicity, (9) is written with the additional assumption that $v \ll c$. It is easiest at several points in this preliminary analysis to assume that the interesting behavior occurs at low speeds, but that assumption can be removed in a more thorough investigation of these phenomena.

The rate at which energy is dissipated on the fracture surface is $\eta(\partial \dot{u}/\partial x)^2$. Let us assume that all of this energy is converted to heat. To obtain a rough estimate of the temperature in the cohesive zone, first use (8) and (9) to compute the average rate at which heat is generated per unit length of the crack in the region $0 < x < l$. Then assume a linear cooling mechanism in which the rate of heat lost from the crack surface is $\gamma \Delta T$, where ΔT is the incremental (above ambient) temperature and γ is a constant. If the crack is moving slowly so that there are no advective effects, we may compute ΔT by equating heat generated to heat lost. The result is

$$\Delta T \approx \frac{K_\infty^2}{3\gamma}\left(\frac{\mu^3 v}{\eta}\right)^{1/2}. \tag{10}$$

Next, as suggested above, let us assume that the dominant temperature dependence in $K_{\rm eff}$ is carried by the yield strain σ_y/μ, which appears as the factor in (5) with the largest exponent. Denote the temperature derivative of $\ln(\sigma_y/\mu)$ by the symbol B. If B is positive, $K_{\rm eff}$ increases with temperature and, according to (4), v/c decreases. For present purposes, we assume that we are dealing only with relatively small changes in temperature so that we need to keep only the first-order term in a Taylor expansion of $\ln(\sigma_y/\mu)$ in powers of ΔT. The correspondingly small change in $K_{\rm eff}$, however, is amplified in (4) because $K_{\rm eff}$ is raised to the twelfth power. Thus, for $v \ll c$, (4) becomes

$$\frac{v}{c} \approx \left(\frac{K_\infty}{K_0}\right)^{12} \exp\left[-8\left(\frac{K_\infty}{K_T}\right)^2\left(\frac{v}{c}\right)^{1/2}\right], \tag{11}$$

where K_0 is the value of $K_{\rm eff}$ at the ambient temperature and $K_T^2 = (3\gamma\eta^{1/2}/B\mu^{3/2}c^{1/2})$. If we take K_T to be approximately a constant, then we can easily solve (11) for v/c as a function of K_∞. If $K_T \lesssim K_0$, the answer is that v/c initially grows like $(K_\infty/K_0)^{12}$ as before, but reaches a maximum value of $0.1449(K_T/K_0)^3$ at $K_\infty = 1.4036 \times (K_0^3 K_T)^{1/4}$. For larger values of K_∞, v/c decreases, but the asymptotic behavior predicted by (11) ($v/c \sim K_\infty^{-4}$) occurs beyond the range of the first-order Taylor approximation. In contrast, for large values of K_T, the crack accelerates to $v \cong c$ at $K_\infty = K_0$, and the thermal effect is inoperative.

Note that if this thermal feedback effect were actually to occur, it would produce a maximum velocity that is independent of the size or geometry of the system. In addition, the decrease in velocity at large applied stress might be an indication of some sort of dynamic instability. The analysis leading to (11), however, is far too speculative to be a firm basis for such predictions. In (8) and (9), we have ignored the possibility that the crack tip deforms in response to local heating. We also have considered only variations in the speed of rectilinear motion and have ignored the possibility that the velocity-limiting instability might involve oscillation in the direction of crack growth, as seems to be seen experimentally [2,6]. Thus, the thermal mechanism outlined here should be understood as being no more than a suggestion of one kind of interesting behavior that might occur in this class of dynamical models.

In conclusion, the model introduced here exhibits a rich variety of physically interesting behaviors, most of which remain to be explored in detail at the time this paper is being written. The most important unanswered question is whether this kind of model is just a mathematical curiosity, or whether it might be realistic enough to be useful. Phenomena such as the onset or arrest of fracture generally are discussed in terms of detailed physical mechanisms such as emission and motion of dislocations, atomic rearrangements at crack tips, formation of shear bands, and the like. If such mechanisms could be incorporated into a small number of phenomenological parameters such as the "viscosity" η, we might gain a powerful new tool for studying the dynamics of fracture.

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[1] Useful general references are M. F. Kanninen and C. H. Popelar, *Advanced Fracture Mechanics* (Oxford Univ. Press, New York, 1985); L. B. Freund, *Dynamic Fracture Mechanics* (Cambridge Univ. Press, New York, 1990).
[2] J. Fineberg, S. P. Gross, M. Marder, and H. L. Swinney, Phys. Rev. Lett. **67**, 457 (1991).
[3] M. Barber, J. Donley, and J. S. Langer, Phys. Rev. A **40**, 366 (1989).
[4] J. S. Langer, Phys. Rev. A **46**, 3123 (1992).
[5] J. S. Langer and H. Nakanishi, Phys. Rev. E (to be published).
[6] A. Yuse and M. Sano, Nature (London) **362**, 329 (1993).