Hydrodynamic Theory of Electromagnetic Fields in Continuous Media

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The hydrodynamic Maxwell equations of continuous media, valid for large amplitudes and small frequencies, are derived. As any hydrodynamic theory, these equations are universal, nonlinear, and irreversible. Consequences for dielectric liquids, ferrofluids, superconductors, and nematic liquid crystals are briefly outlined.

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The Maxwell equations of continuous media,

$$\dot{\mathbf{B}}=-\nabla\times\mathbf{E},\quad \dot{\mathbf{D}}=\nabla\times\mathbf{H}-\mathbf{j}_e,\quad \nabla\cdot\mathbf{B}=0,\quad \nabla\cdot\mathbf{D}=\rho_e\,, \tag{1}$$

are rigorous, reversible, and incomplete. It was Lorentz [1] who taught us how to understand them. Two steps are involved: First, one starts from the microscopic Maxwell equations containing only **B** and **E**, and divides [2] the charge density into ρ_e and $\nabla\cdot\mathbf{P}$, and the current density into $\mathbf{j}_e$, $\dot{\mathbf{P}}$, and $\nabla\times\mathbf{M}$. The polarization **P** and magnetization **M**, which subsume all charges and currents that vary on the smaller, atomistic scales, are then added to **B** and **E**: $\mathbf{H}=\mathbf{B}/\mu_0-\mathbf{M}$, $\mathbf{D}=\varepsilon_0\mathbf{E}+\mathbf{P}$, while preserving the structure of the Maxwell equations. This alone already yields Eqs. (1), which are therefore just as rigorous as the microscopic Maxwell equations and very much incomplete, since they are valid irrespective of how **P** and **M** are defined. The reversibility of Eqs. (1), a hallmark of microscopic physics, is manifest: Each equation contains only terms of the same parity under time reversal. To complete the description, one needs as the second step: (i) a clear prescription for the above division and (ii) the time dependence of the resultant **M** and **P**. Generally, this entails the solution of a full-blown many-body problem; linear response theory, certainly feasible, is confined to weak fields; simplest is the quasistatic approximation that takes **P** and **M** as functions of **E** and **B**, respectively.

For comparison, consider the hydrodynamic equations of an isotropic liquid, especially the Navier-Stokes equation [3]. These equations are universal, closed, irreversible, and nonlinear. They are independent of microscopic interaction, valid for (say) helium gas, water, and liquid metal. The number of fields and equations of motion are the same, no microscopic or nonhydrodynamic information (such as on $\dot{\mathbf{P}}$ or $\dot{\mathbf{M}}$ in the previous case) is needed for the prediction of the fields' time evolution. Dissipative, irreversible terms account for viscosity and heat diffusion, produce entropy, and restore equilibrium. Finally, shock waves, convective instabilities, and other nonlinear, large amplitude phenomena are well accounted for. The price for these nice features is a confined range of validity: low frequency and small wave vector, $\omega\tau, k\xi \ll 1$. (In practice, of course, the time τ for establishing local equilibria and the correlation length ξ that defines their minimum spatial extent vary over orders of magnitude for different systems at different temperatures. Hence the price one pays is only occasionally painful.)

A set of *hydrodynamic* Maxwell equations can be derived that possesses all the above features. They are

$$\begin{aligned}
&\dot{\mathbf{D}}=\nabla\times\mathbf{H}-(\hat{\sigma}\mathbf{E}_0+\rho_e\mathbf{v}+\hat{\gamma}\nabla T)-\nabla\times(\hat{\alpha}\nabla\times\mathbf{E}_0)\,,\\
&\dot{\mathbf{B}}=-\nabla\times\mathbf{E}-\nabla\times(\hat{\beta}\nabla\times\mathbf{H}_0)\,,\\
&\nabla\cdot\mathbf{B}=0,\quad \nabla\cdot\mathbf{D}=\rho_e\,,
\end{aligned}\tag{2}$$

where **E** and **H** are functions (in general nonlinear) of the thermodynamic variables [4], especially **D** and **B**. ($\mathbf{H}_0$ and $\mathbf{E}_0$ are the respective fields in the local rest frame, moving with the fluid velocity **v**.) Equations (2) are valid for any *isotropic system in local equilibrium* (isotropic if without fields). Modifications occur only for systems with spontaneously broken symmetries such as superconductors and nematic liquid crystals. (We shall consider them later.) Equations (2) are irreversible, the dissipative terms being those preceded by the tensors of kinetic coefficients $\hat{\alpha}$, $\hat{\beta}$, $\hat{\gamma}$, and $\hat{\sigma}$. While $\hat{\sigma}$ is obviously the electric conductivity (both Ohm and Hall), and $\hat{\gamma}$ the Peltier tensor of the thermoelectric effect, $\hat{\alpha}$ and $\hat{\beta}$ are new. They precede the only dissipative terms in a nonconducting medium. *Without them, Eqs. (2) cannot restore equilibrium there.* We shall later calculate the values of $\hat{\alpha}$ and $\hat{\beta}$, in a simple model, and see that they are connected to **M** and **P** not being quite in equilibrium. With two equations of motion accounting for two fields, **D** and **B**, Eqs. (2) are obviously closed. Last but not least, Eqs. (2) are nonlinear and describe arbitrary field strengths.

The derivation of Eqs. (2) closely resembles derivations of other hydrodynamic equations. In fact, for a stationary system ($\mathbf{v}\equiv 0$) and without coupling to the temperature ($\hat{\gamma}\equiv 0$), it is amazingly simple. We start from the thermodynamic relation for the energy density [2] $d\varepsilon = T\,ds+\mathbf{E}\cdot d\mathbf{D}+\mathbf{H}\cdot d\mathbf{B}$, which unambiguously defines $\mathbf{E}\equiv\partial\varepsilon/\partial\mathbf{D}$ and $\mathbf{H}\equiv\partial\varepsilon/\partial\mathbf{B}$, in the same sense temperature is defined thermodynamically, as an equation of state. (The mass density is quite irrelevant in this context, hence $\mu\,d\rho$ is suppressed.) The variables themselves are in turn defined by the two nontemporal Maxwell equations, $\nabla\cdot\mathbf{D}=\rho_e$ and $\mathbf{B}=\nabla\times\mathbf{A}$. As long as these two equations

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are satisfied, any $\mathbf{D}(\mathbf{r})$ and $\mathbf{B}(\mathbf{r})$ fields are permissible. Only in equilibrium do $\mathbf{D}$ and $\mathbf{B}$ in addition satisfy $\mathbf{E}=0$ in a conductor, $\nabla\times\mathbf{E}=0$ in a dielectric medium, and $\nabla\times\mathbf{H}=0$ generally. (The current of the usual static equation is missing here, since it would produce entropy. It appears later in the stationary solution, or in the equilibrium condition of the superconductors.) These equilibrium conditions result from minimizing the total energy $\int\varepsilon d^3r$ of a given volume, with respect to variations in $\mathbf{D}$ and $\mathbf{B}$ with the appropriate constraints [2]: $\int\rho_e d^3r$ given within the connected volume of a conductor; $\nabla\cdot\mathbf{D}$ given for every point of a dielectric medium and $\nabla\cdot\mathbf{B}$ vanishing everywhere. Now the bold step that becomes less daring the more one thinks about it. Outside equilibrium, the $\mathbf{D}$ and $\mathbf{B}$ fields produce entropy, the local rate $T\dot{s}=R$ of which can (close to equilibrium) be expanded in the small parameters $\mathbf{E}$, $\nabla\times\mathbf{H}$, and $\nabla\times\mathbf{E}$,

$$R=\sigma_{ij}E_iE_j+\beta_{ij}(\nabla\times\mathbf{H})_i(\nabla\times\mathbf{H})_j+\alpha_{ij}(\nabla\times\mathbf{E})_i(\nabla\times\mathbf{E})_j\,, \tag{3}$$

just as one would expand R in ∇_iT or $\nabla_iv_j+\nabla_jv_i$. ($\hat{\sigma}$ and $\hat{\alpha}$ are not necessarily mutually exclusive: A bad conductor may well reach $\nabla\times\mathbf{E}=0$ much more quickly than $\mathbf{E}=0$. Then the first term in R represents a small perturbation, which nevertheless should be included for its long time effect. In a good conductor, the third term should not be important.) Since the system is isotropic if without field, the three tensors are of the structure $\alpha\delta_{ij}+\alpha_EE_iE_j$ for $H\to0$, and $\alpha\delta_{ij}+\alpha_HH_iH_j+\alpha_\varepsilon\varepsilon_{ijk}H_k$ for $E\to0$. All coefficients are functions of the thermodynamic variables, especially the fields [5]. α_ε is reactive, i.e., it only contributes to b_i below, not to R; the same holds for β_ε and σ_ε, the latter being the Hall conductance. [It is not difficult to write down the higher order terms in Eq. (3) if necessary. Because of the proliferation of transport coefficients, however, this is not an overall useful exercise.] While producing entropy, $\mathbf{D}$ and $\mathbf{B}=\nabla\times\mathbf{A}$ also evolve with time, and their equations of motion can be quite generally taken to be

$$\dot{D}_i=\delta\varepsilon/\delta A_i-j_i^e+\nabla_jb_{ij},\quad \dot{A}_i=-\delta\varepsilon/\delta D_i-a_i\,, \tag{4}$$

where the terms j_i^e, b_{ij}, and a_i are yet to be determined. With ε being a conserved quantity, we may write $-\nabla\cdot\mathbf{Q}=\dot{\varepsilon}=T\dot{s}+\mathbf{E}\cdot\dot{\mathbf{D}}+\mathbf{H}\cdot\nabla\times\dot{\mathbf{A}}$, insert Eqs. (4), and insist on being able to write the right-hand side as a total divergence. This results in

$$Q=(\mathbf{E}+\mathbf{a})\times\mathbf{H}+\mathbf{E}\times\mathbf{b} \tag{5}$$

and

$$R=\mathbf{a}\cdot(\nabla\times\mathbf{H})-\mathbf{b}\cdot(\nabla\times\mathbf{E})+S_{ij}E_{ij}+\mathbf{j}_e\cdot\mathbf{E}\,, \tag{6}$$

where

$$b_{ij}=b_k\varepsilon_{kij}+S_{ij},\quad S_{ij}=S_{ji}\,,$$

and

$$E_{ij}=\tfrac{1}{2}(\nabla_iE_j+\nabla_jE_i)\,.$$

Comparing Eq. (6) with (3), we find $j_i^e=\sigma_{ij}E_j$, $a_i=\beta_{ij}(\nabla\times\mathbf{H})_j$, $b_i=-\alpha_{ij}(\nabla\times\mathbf{E})_j$, and $S_{ij}=0$, which brings us to Eqs. (2) for $\mathbf{v}=0$, $\hat{\gamma}=0$.

There is an intriguing analogy worth exploring. Hydrodynamic variables frequently form canonically conjugate pairs, of which usually one is a conserved quantity and the other a broken symmetry variable. An example is the pair: density ρ and phase φ in superfluids [3]. The energy depends on $\nabla\varphi$, and the reactive, nonconvective dynamics is given by $\dot{\rho}=\delta\varepsilon/\delta\varphi=-\nabla(\partial\varepsilon/\partial\nabla\varphi)$, $\dot{\varphi}=-\partial\varepsilon/\partial\rho=-\mu$. Another example is the pair: momentum density $\mathbf{g}$ and displacement vector $\mathbf{u}$ in elasticity theory [6]. For small $\mathbf{u}$, the energy depends on $u_{ij}=(\nabla_iu_j+\nabla_ju_i)/2$, while the dynamics is again $\dot{g}_i=-\delta\varepsilon/\delta u_i=\nabla_j\partial\varepsilon/\partial u_{ij}$, $\dot{u}_i=\partial\varepsilon/\partial g_i=v_i$. Compare these with the pair $\mathbf{D}$, $\mathbf{A}$. Obviously, the energy again depends on the spatial derivative of $\mathbf{A}$, and the reactive dynamics conforms to the canonical form $\dot{\mathbf{D}}=\delta\varepsilon/\delta\mathbf{A}=\nabla\times\mathbf{H}$, $\dot{\mathbf{A}}=-\partial\varepsilon/\partial\mathbf{D}=-\mathbf{E}$. Two remarks: (i) The special gauge of $\dot{\mathbf{A}}=-\mathbf{E}$ was chosen to highlight its close analogy to $\dot{\varphi}=-\mu$ or $\dot{\mathbf{u}}=\mathbf{v}$. The derivation leading to Eqs. (5),(6) could of course have been performed without a gauge choice. (ii) The cross terms $\mathbf{a}=\hat{\eta}\mathbf{E}+\cdots$, $\mathbf{j}^e=-\hat{\eta}\nabla\times\mathbf{H}+\cdots$ are indeed allowed by symmetry, but would result in $\dot{\mathbf{A}}=-(1+\hat{\eta})\mathbf{E}+\cdots$, which does not comply with the canonical relations and contradicts Eqs. (1) in the linear response regime. Therefore, $\hat{\eta}=0$. [If $\hat{\eta}$ were a constant scalar, a rescaling $\mathbf{A}/(1+\eta)\to\mathbf{A}$ would render it unnoticeable.]

Except σE, a sink term accounting for relaxation, all other terms in $\dot{\mathbf{D}}$ and $\dot{\mathbf{B}}$ of Eqs. (2) are currents, accounting for "transport." To better understand the physics represented by $\hat{\alpha}$ and $\hat{\beta}$, we may consider a simple model of a paramagnetic dielectric medium with $\hat{\sigma}=0$. Assuming $\dot{\mathbf{M}}=-(M-M_{\rm eq})/\tau_M$, and $\dot{P}=-(P-P_{\rm eq})/\tau_P$, where $M_{\rm eq}$ and $P_{\rm eq}$ are the respective local and instantaneous equilibrium values, a quick calculation for τ_P, $\tau_M\ll\omega^{-1}$ yields

$$\beta\varepsilon_0=\tau_P(1-\varepsilon_0\partial^2\varepsilon/\partial D^2),\quad \alpha\mu_0=\tau_M(1-\mu_0\partial^2\varepsilon/\partial B^2)\,, \tag{7}$$

while the other coefficients vanish. Note the curious cross relationship: β in $\dot{\mathbf{B}}$ depends on τ_P, and α in $\dot{\mathbf{D}}$ depends on τ_M. Of course, $\mathbf{M}$ and $\dot{\mathbf{P}}$ are in general much more complicated, but this only alters the value of α and β, not the hydrodynamic terms they precede. These terms, neglected up until now, are qualitatively important, since they are the very ones that restore equilibrium with respect to $\nabla\times\mathbf{E}=0$ and $\nabla\times\mathbf{H}=0$. Therefore, hydrodynamic calculations (such as in ferrofluids or nematics) employing Eqs. (2) without α and β would indeed lead to blatantly nonsensical results and is not usually done. Prevalent are hydrodynamic calculations in which the equilibrium conditions $\nabla\times\mathbf{H}=\nabla\times\mathbf{E}=0$ are imposed, with the notion that these are always much more quickly achieved than the other equilibrium conditions such as constant temperature. However, $\mathbf{M}$ and $\mathbf{P}$ lag behind external perturbations as any internal, nonhydrodynamic variables do and produce entropy on the hydrodynamic level. There is no reason to believe that their relaxation times are always much smaller. In fact, it is incorrect to assume $\nabla\times\mathbf{E}=\nabla\times\mathbf{H}=0$ even for stationary, nonequilibrium situations of strictly vanishing frequencies, cf. the static, nonpotential electric field of Eq. (8) below.

What are the ramifications of the hydrodynamic Maxwell equations? First and foremost, **D** and **B** are now treated on equal footing as the other thermo and hydrodynamic variables. Therefore, coupling among them becomes important. Before we embark on this subject, however, in the manageable form of a quick dip into three illustrative examples, we shall first discuss three simple consequences of Eqs. (2) while assuming $\hat{\alpha}$, $\hat{\beta}$, and $\hat{\sigma}$ to be scalars, and $\mathbf{v}$, $\hat{\gamma}=0$.

(i) The stationary solution ($\dot{\mathbf{D}}=\dot{\mathbf{B}}=0$) for $\alpha=0$ is

$$\nabla\times\mathbf{H}=\sigma\mathbf{E},\quad \nabla\times\mathbf{E}=\mathbf{E}\times\nabla(\beta\sigma)/(1+\beta\sigma)\,. \tag{8}$$

If $\nabla(\beta\sigma)=0$, these are familiar static Maxwell equations. In the presence of a temperature or pressure gradient, however, because β and σ are in general functions of these variables, **E** is nonpotential. [There is no contradiction to Eqs. (1), where **E** is *defined* as $-\dot{\mathbf{A}}$, rather than $\partial\varepsilon/\partial\mathbf{D}$. Then of course $\nabla\times\mathbf{E}=0$ if $\dot{\mathbf{B}}=0$.)

(ii) In the opposite limit (ideal dielectrics, $\sigma=0$), the higher order gradient terms render the stationary solution more complicated, giving rise to fields decaying exponentially near the surface. They are very similar to the hydrodynamic *sq* modes [7]. In the same system, propagation of electromagnetic waves (at low frequency and small amplitude) is described by

$$\omega^2/q^2=(\partial^2\varepsilon/\partial B^2)(\partial^2\varepsilon/\partial D^2)$$
$$-i\omega(\beta\partial^2\varepsilon/\partial B^2+\alpha\partial^2\varepsilon/\partial D^2)\,. \tag{9}$$

Given the structure of Eqs. (2), with the conductivity vanishing, the soundlike damping is hardly surprising. But this expression is valid in the presence of arbitrarily large static fields and does provide a way to measure (or calculate) α and β.

Now we study the coupling to the other hydrodynamic variables. The hydrodynamic theory of an isotropic polarizable and magnetizable liquid was recently derived, whereby a number of old and controversial questions (such as what the total momentum or what the pressure is) had to be settled [8]. However, adhering to orthodox faith [9], **b** and **a** of Eqs. (4) were taken to be zero. Rederiving the liquid's hydrodynamic theory including **a** and **b**, the bulk of the equations in Ref. [8] is found unchanged, except the entropy production, which becomes

$$R=\mathbf{f}\cdot\nabla T\cdots+\mathbf{a}\cdot(\nabla\times\mathbf{H}_0)-\mathbf{b}\cdot(\nabla\times\mathbf{E}_0)$$
$$+(\mathbf{j}^e-\rho_e\mathbf{v})\cdot\mathbf{E}_0\,, \tag{10}$$

where **f** is the entropy current; the other thermodynamic forces are not displayed here. The dissipative terms of Eqs. (2) follow directly from Eq. (10), a manifestly Galilean invariant quantity. (The cross terms $\mathbf{a}=\hat{\xi}\nabla T+\cdots$, $\mathbf{f}=-\hat{\xi}\nabla\times\mathbf{H}+\cdots$ are permitted by symmetry. Again, if $\hat{\xi}$ is a constant scalar, both terms represent an unnoticeable gauge transformation; otherwise, they are of higher order. Hence $\hat{\xi}=0$ for simpler display.) The equations of Ref. [8], amended with Eqs. (2) and (10), are the proper and rigorous hydrodynamic theory of ferrofluids [10]. In comparison, the usual approach (consisting of the static Maxwell equations, a postulated equation of motion for the magnetization, and the complete neglect of the polarization [11]) seems deficient, especially in light of Eqs. (7). A detailed discussion will be published elsewhere.

We turn our attention briefly to superconductors. The most important modification here is the additional dependence of the energy on the gauge invariant combination $(\hbar/2e)\nabla\varphi-\mathbf{A}$, φ being the phase, spontaneously broken in superconductors. As a result, $\delta\varepsilon/\delta\mathbf{A}=\nabla\times(\partial\varepsilon/\partial\nabla\times\mathbf{A})-\partial\varepsilon/\partial\mathbf{A}=\nabla\times\mathbf{H}-\mathbf{j}_s=0$ is the new equilibrium condition, where $\mathbf{j}_s$ denotes the supercurrent. Naturally, $\nabla\times\mathbf{H}-\mathbf{j}_s$ is now the thermodynamic force, and R a quadratic function of it. Hence $\nabla\times\mathbf{H}-\mathbf{j}_s$ substitutes $\nabla\times\mathbf{H}$, both in $\dot{\mathbf{D}}$ and $\dot{\mathbf{B}}$ of the hydrodynamic Maxwell equations, Eqs. (2).

Nematic liquid crystals are interesting in this context because they permit very direct dissipative couplings such as between the two thermodynamic forces, $\nabla\times\mathbf{E}_0$ and $\mathbf{n}\times\boldsymbol{\psi}$ (where $\boldsymbol{\psi}\equiv\delta\varepsilon/\delta\mathbf{n}$ is the "molecular field" [6] of the director **n**),

$$\begin{aligned} d\mathbf{n}/dt&=\cdots\mathbf{n}\times(\hat{\gamma}\mathbf{n}\times\boldsymbol{\psi}+\hat{\eta}\nabla\times\mathbf{E}_0)\,,\\ \dot{\mathbf{D}}&=\cdots-\nabla\times(\hat{\alpha}\nabla\times\mathbf{E}_0+\hat{\eta}\mathbf{n}\times\boldsymbol{\psi})\,. \end{aligned} \tag{11}$$

[The tensors are now anisotropic even in the weak field limit, e.g., $\alpha_{ij}=\alpha_\perp(\delta_{ij}-n_in_j)+\alpha_\parallel n_in_j$.] Note that, due to the spontaneously broken rotational symmetry, the Maxwell equations are modified and include typically nematic terms. These are the proper equations to describe the field-driven convective instabilities of nematic liquids [12].

Finally and briefly, the boundary conditions: As a result of the second order spatial derivatives in Eqs. (2), ten rather than six boundary conditions are needed to completely specify any solution. They can be obtained in a standard procedure that has been recently developed for, and applied with success to, other hydrodynamic systems [7]. These include isotropic liquid as well as systems with broken symmetries such as superfluid, antiferromagnets and liquid crystals.

In summary, a set of irreversible, closed and nonlinear Maxwell equations is derived according to the hydrodynamic concept. Two additional thermodynamic forces are identified, $\nabla\times\mathbf{H}_0$ and $\nabla\times\mathbf{E}_0$. They give rise to dissipative terms and represent the mechanisms for electromagnetic fields to restore equilibrium in nonconducting media. Two effects are considered: a nonpotential static E field in the presence of a temperature or pressure gradient, and the soundlike damping of the electromagnetic waves. Many more consequences are expected, especially for ferrofluids, superconductors, and nematics, where the Maxwell equations are explicitly modified.

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[4] Identifying $\mathbf{D}_1=\mathbf{D}_2$, $\mathbf{B}_1=\mathbf{B}_2$ (where the subscripts refer to the number of the displayed equations), we find $\mathbf{H}_1=\mathbf{H}_2-\hat{\alpha}\nabla\times\mathbf{E}_2$, $\mathbf{E}_1=\mathbf{E}_2+\hat{\beta}\nabla\times\mathbf{H}_2$ in a system at rest. These two expressions define $\mathbf{H}_1$ and $\mathbf{E}_1$ in terms of $\mathbf{D}_1$ and $\mathbf{B}_1$ in the hydrodynamic regime, and renders Eqs. (1) complete. (Strictly speaking, however, only the divergence of **D** and **B** may be identified, from the two nontemporal Maxwell equations.)

[5] Generally, the system is biaxial and every tensor is characterized by six dissipative and three reactive coefficients. Note that E serves both as a thermodynamic variable and a generalized force. In dielectrics, the first role dominates, in conductors, the second does.

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