Fractal-Like Quasienergy Spectrum in the Fermi-Ulam Model

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We prove the existence (as $\hbar \to 0$) of the singular continuous ("Cantor-like") spectrum for the Fermi-Ulam model on a torus for a certain phase-space domain and an irrational (diophantine) ratio of the minimum distance between the walls to the oscillation amplitude of the moving wall. Varied scenario of the stabilizing and destabilizing spectral transitions are observed in the limit of vanishing Planck's constant, and their relation with the qualitative changes in the quasienergy eigenstates is brought out.

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Classically chaotic behavior and its manifestations has been a subject of extensive research in recent times [1]. Nonlinear systems driven by an external, time-dependent perturbation serve as convenient models where such behaviors are expected to manifest. Gradually, it is becoming quite clear that classical chaos is almost always accompanied by some signature that reveals a similar structure on a progressively finer scale, i.e., fractal sets [2]. Based on the exact results on a periodically kicked quantum rotator, it was conjectured [3] that the classical chaotic behavior is reflected (at least partially) in the singular continuity of the quasienergy spectrum. In view of the correspondence principle, we call a logical juxtaposition of these statements the Casati-Gutzwiller conjecture.

It is well known that the dynamics behavior of time-dependent systems is intimately related to the spectral properties of the evolution operator in the classical (via diffeomorphisms) [4] and the quantal (via, e.g., Floquet operator) formalisms [5]. A transition among different types of spectra [pure point (PP), absolutely continuous (AC), singular continuous (SC)] of the quantal evolution operator is interpreted as a "spectral transition" or as an occurrence of a quantum instability. Some well-known examples of such instabilities are the Peierls instability [6] and the subthreshold microwave ionization of hydrogen atoms [7]. The most popular model examples of relevance to us here are the kicked rotator and the Fermi-Ulam model (or the Fermi accelerator). For the kicked rotator, it was rigorously proven that for a generic choice of a driving potential, there does exist a continuous component in the quasienergy spectrum [8]. However, it was not clear whether this is AC or SC; also there remained the difficulty of quantifying the condition of genericity. For the kicked rotator subjected to a separable, rank one perturbation (introduced in Ref. [9]), it was proven that the quasienergy spectrum is purely singularly continuous [10] under certain conditions. Noting that the SC spectrum resides on a Cantor-like set, it is of utmost importance to prove its existence rigorously in order to avoid the uncontrollable numerical errors that destroy the nonrecurrent behavior, yielding to false quasiperiodicity.

Diffusive growth of the momentum and phase was shown in the kicked rotator (KR) [11,12] and the Fermi-Ulam model (FUM) [13], respectively, indicating that diffusion (continuous spectrum) [14] is a hallmark of chaos (rigorously speaking, a continuous spectrum simply implies weak mixing). Moreover, both these models were mapped onto the Lloyd model of disordered solids in an infinite [15] and a finite [13] lattice of one dimension. This implies the localization of quasienergy eigenstates, suppressing thereby the classically weak-mixing behavior. These studies present the following picture: up to a time, t^* (break time), the quantal system mimics (pseudo-) diffusive dynamics where the wave packet keeps spreading and the mean-square phase or momentum increases linearly with kicks, after which the manifestly quasiperiodic motion sets in. Following Chirikov [16], this phenomenon is termed as a finite-time statistical relaxation in the discrete spectrum. Let us interpret this change in the dynamical behavior in terms of changes in the nature of the spectrum of the evolution operator. Evolution of a kicked system between two successive kicks is governed by the Floquet operator, $\hat{F}$. For larger times, the evolution is governed by the evolution operator, $\hat{U}=\hat{F}^n$. Thus, although the Floquet operator is time independent, the evolution operator is time dependent. By Ruelle's argument [14], pseudodiffusive dynamics corresponds to a continuous spectrum. On the other hand, quasiperiodic dynamics is related to a PP spectrum. We see, therefore, that a change in dynamics is associated with a qualitative change in the nature of the quasienergy spectrum. Quantum mechanically, for a different potential in the FUM, it was shown that the Δ_3 statistic goes from $L/15$ (Poisson) to $(\pi^{-2}\ln L = 0.007\ldots)$ [Gaussian orthogonal ensemble (GOE)], L being the length of the interval on which the spectral average is computed, as the coupling goes from weak to strong, i.e., as the parameter $4/\hbar\pi$ increases [17]. The Schrödinger equation for the Fermi accelerator (i.e., with Dirichlet boundary conditions at the walls) was treated [18] wherein it turned out that the classical limit of the quantum system is just the classical Fermi accelerator for only those values of

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momentum that correspond to the stable islands in the stochastic sea of the divided phase space. It is worth noting that the Fermi accelerator [19] can be mapped exactly on the time-dependent harmonic oscillator confined between two stationary walls using the generalized canonical transformations. For this confined oscillator, an exact propagator has been derived and it was shown that the Lewis invariant breaks down [20].

In this Letter, we investigate the nature of the spectrum of the evolution operator with some conditions related to irrationality of relevant parameters in the FUM. In particular, our aim is to arrive at a rigorously proven statement about the FUM on a torus in view of the Casati-Gutzwiller conjecture. In the FUM [21], a particle of mass (m set to unity) moves between two infinitely massive walls, one of which is oscillating at a speed, $V_{\rm wall}(t)$ which is a periodic sawtooth function. The wall oscillation period is T and the maximum speed of the wall is $V_0/4$. The speed of the particle is denoted by v which is measured in terms of the dimensionless variable, $u=4v/V_0$. The time t is measured in terms of the dimensionless variable $\phi=\{t/T\}$ where $\{\cdots\}$ represents the fractional part. In the considerations addressed in this Letter, we shall put the FUM on the torus, the domains of ϕ and u being $[-\frac{1}{2},\frac{1}{2}]$ and $(1,e]$, e being exponential of unity. The minimum distance between the walls is l and the amplitude of the moving wall is a. As the particle moves, it collides elastically with the walls alternatively (assuming $v\gg V_0/4$). Depending upon whether $\phi\in(0,\frac{1}{2}]$ $\{[-\frac{1}{2},0)\}$, the particle gains (loses) in speed. On an average, the particle accelerates with collisions. The speed and the phase of the particle after $(n+1)$th collision, u_{n+1} and ϕ_{n+1}, respectively, can be expressed in terms of u_n and ϕ_n by [22]

$$\begin{aligned}\phi_{n+1}&=\phi_n+M/u_{(n+1)'}\bmod 1\,,\\ u_{n+1}&=u_n+\phi_n\,,\end{aligned}\tag{1}$$

where $M=1/16a$. It was shown [19] that this map exhibits chaotic behavior in the (ϕ,u) plane when $u<\sqrt{M}/2$, and, there exist smooth invariant curves when $u>\sqrt{M}/2$.

It is evident from (1) that ϕ and u are canonically conjugate phase space variables. As linear operators, they satisfy the commutation relation. With a description where the Hamiltonian is quadratic in ϕ [13], one can prove that (1) with (ϕ,u) replaced by $(\hat\phi,\hat u)$ are just the discrete-time Heisenberg equations. Since we are interested in analyzing the nature of the spectrum, it is convenient to write the Hamiltonian as

$$\hat H(\hat\phi,\hat u)=\alpha\hat\phi+\hat V(\hat u)\sum_{n=-\infty}^{\infty}\delta(t-n)\,,\tag{2}$$

$\alpha=8a/l$ and $\hat V(\hat u)=\ln(1/\hat u)+(e-1)^{-1}$. Since $a\ll l$, $\alpha\in[0,1]$. For our purpose, we choose α to be an irrational, diophantine number of order σ ($\sigma\geq 2$), i.e., there exists $\gamma>0$ such that $|\alpha-p/q|\geq\gamma q^{-\sigma}$ for all p/q belonging to the set of rational numbers. If $\hat F(\alpha)$ is the Floquet operator and $\psi(u)$ is a wave function satisfying

$$\hat F(\alpha)\Psi(u)=\exp[iV(u)]\psi(u-\Gamma\alpha)\,,\tag{3}$$

where $\Gamma=e-(1+\delta)$, δ, however small, is a positive definite number. Since the FUM is on a torus, it trivially follows that the eigenvalues of ϕ are $2\pi n\hbar/\Gamma$ ($=\lambda_n$) where $n=-n_{\max},\ldots,0,\ldots,n_{\max}$; $n_{\max}=\Gamma/4\pi\hbar$ (since $\phi\in[-\frac{1}{2},\frac{1}{2}]$). It must be noted that in the semiclassical limit ($\hbar\to 0$) the number $n_{\max}$ will increase ($\to\infty$). Within the semiclassical approximation (i.e., letting $n_{\max}$ tend to infinity, and setting h to unity for convenience), one can verify that

$$\begin{aligned}&V(u)=\sum_n V_n\exp(i\lambda_n u)\,,\\ &V_0=\Gamma^{-1}\int_{1+\delta}^{e}du\,V(u)=0\quad\text{as }\delta\to 0\ (\text{recall that }u\in(1,e])\,.\end{aligned}\tag{4}$$

The Fourier coefficients, V_n, can be expressed as

$$\begin{aligned}V_n=(\Gamma\lambda_n)^{-1}(&\{-\sin(\lambda_n e)+\delta\sin[\lambda_n(1+\delta)]+\mathrm{si}(\lambda_n e)-\mathrm{si}[\lambda_n(1+\delta)]+\Gamma^{-1}\sin(\lambda_n e)-\Gamma^{-1}\sin[\lambda_n(1+\delta)]\}\\ &+i\{-\cos(\lambda_n e)+\delta\cos[\lambda_n(1+\delta)]+\mathrm{ci}(\lambda_n e)-\mathrm{ci}[\lambda_n(1+\delta)]+\Gamma^{-1}\cos(\lambda_n e)-\Gamma^{-1}\cos[\lambda_n(1+\delta)]\})\,,\end{aligned}\tag{5}$$

where $\mathrm{ci}(x)$ and $\mathrm{si}(x)$ are cosine and sine integrals defined by [23]

$$\mathrm{ci}(x)=-\int_x^\infty dt\,t^{-1}\cos t,\ \ \mathrm{si}(x)=-\int_x^\infty dt\,t^{-1}\sin t\,.\tag{6}$$

Notice that, as $x\to\infty$, $\mathrm{ci}(x)$ and $\mathrm{si}(x)$ tend to zero. Since the sequence $\{V_n\}$ is convergent, it can be shown that

$$0<\liminf_{|n|\to\infty}|V_n|^{1/|n|}=\limsup_{|n|\to\infty}|V_n|^{1/|n|}\leq 1\tag{7}$$

as $\delta\to 0$ and $n_{\max}$ increases. Furthermore, let us define the function

$$L(\alpha)=\limsup_{|n|\to\infty}|\lambda_n|^{-1}\ln[|\csc(\Gamma\alpha\lambda_n/2)|]\,.\tag{8}$$

It should be noted that $L(\alpha)$ is zero for the set of α's with full Lebesque measure as it contains the diophantine numbers, and, $L(\alpha)$ is infinite for the set of α's that is dense G_δ set in $\mathbb{R}$. In the present case, it is quite clear that $L(\alpha)$ is greater than zero owing to the condition on α. Equation (7) and positivity of $L(\alpha)$ enables us to use the Ruelle-Amrein-Georgescu-Enss (RAGE) theorem [24]. That is, with these conditions, it can be shown that

$$\liminf_{n\to\infty} N^{-1}\sum_{n=0}^{N-1} |\langle e_m|\hat{F}^n|e_m\rangle|^2=0 \tag{9}$$

for all $m\in\mathbb{Z}$: where $e_m=\exp(i\lambda_m u)$. This implies that the pure point spectrum is empty [25]. Hence, the dynamics is not quasiperiodic.

Moreover, since the quantity

$$\Gamma\,\mathrm{Var}(V)=\int_{1+\delta}^{e}|dV/du|\,du<1\,, \tag{10}$$

we can apply Bellissard's theorem [5]. We know that

$$\hat{F}^n\psi(u)=\exp\left(i\Gamma\sum_{l=0}^{n-1}\hat{V}(u-\Gamma\alpha l)\right)\psi(u-\Gamma\alpha\lambda_n)\,. \tag{11}$$

Let p/q be a rational approximant of α such that

$$|\alpha-p/q|\le q^{-2}\,. \tag{12}$$

Then, it can be shown that

$$\limsup_{q\to\infty}|\langle\psi|\hat{F}^q|\psi\rangle|\ge 1-\Gamma\,\mathrm{Var}(V)>0\,. \tag{13}$$

In view of (10), it follows that the absolutely continuous spectrum is empty. Combining this result with the fact proven above about the absence of the pure point spectrum, it is established [26] that the Floquet operator, F, has a purely singularly continuous spectrum [27]. Also, it follows that the spectrum of the evolution operator, $\hat{U}=\hat{F}^n$, is purely singularly continuous. Let us recall that we have shown this result only in the semiclassical limit.

As mentioned above [13], from the classical calculations, it is clear that the diffeomorphism governing the Hamiltonian flow in this system possesses an absolutely continuous spectrum. This trend is reflected in the quantal domains in the continuous spectrum of the Floquet operator up to the time, t^*. Because of the analogy that this system shares with the localization problem, it is clear that after the time, t^*, the spectrum ceases to be continuous. A transition in the spectrum from purely continuous to pure point is an example of a "stabilizing spectral transition" (SST). On the other hand, if α is an irrational (Liouville) number, we get a stability result in the quantal domain. Finally, if α is a diophantine number, then it follows from our arguments that the quasienergy spectrum is SC in the semiclassical region. It must be noted at this juncture that an analytical result of this nature has an enhanced value as the SC spectrum being such a fragile object that numerically one can easily obtain all signatures of a PP spectrum rather spuriously. Thus, in a rather subtle manner, there is persistence of weak mixing (akin to pseudodiffusive dynamics) in semiclassical domain—an instability statement we call a "destabilizing spectral transition" (DST).

In the limit of the Planck's constant approaching zero, we are facing here a novel scenario: up to a time, t^* ($\approx M^2$ [13]), the spectrum of the evolution operator is continuous for all values of $\hbar$, followed by a transition from SST to DST. It is well known that the recurrent and resonance patterns are connected to localized and extended states in the Anderson (or Lloyd) model. Also, it is known that exotic states in the Anderson (or Lloyd) model correspond to a singular continuous spectrum [28], and that these states are related to nonrecurrent, nonoscillatory dynamical patterns. Thus, as $\hbar\to 0$, the qualitative behavior of the states undergoes a transition from extended to localized to exotic.

To summarize, there are enough numerical results to suggest that quantum mechanics appears to be more stable and predictable than classical mechanics. Also there is overwhelming evidence for the existence of fractal sets [29] in the classically nonintegrable, nonlinear systems. We have shown here that a persistence of weak mixing in the quantal domain is reflected in the singular continuity of the quasienergy spectrum. That this finding coexists with the quantal suppression of pseudodiffusion presents a puzzling and an interesting dichotomy in our understanding of a theory of spectral transitions. We believe, based on the results of the kicked rotator and the Fermi-Ulam model (both the systems being examples of generic chaotic dynamical systems), that the Casati-Gutzwiller conjecture is generally valid for periodically driven (1+1)-dimensional, nonlinear systems. Work on other systems is currently in progress; however, detailed discussions and rigorous proofs will appear shortly [20,30].

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