Salient Features of High-Energy Multiparticle Distributions Learned from Exact Solutions of the One-Dimensional Ising Model

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We have derived explicit expressions in the one-dimensional Ising model for multiplicity distributions $P_{\delta\xi}(n)$ and factorial moments $F_q(\delta\xi)$. We identify the salient features of $P_{\delta\xi}(n)$ that lead to scaling, $F_q(\delta\xi)=\tilde{F}_q[F_2]$, and universality. These results compare well with the presently available high-energy data of $\bar{p}p$ and e^+e^- reactions. We point out the important features that should be studied in future higher-energy experiments of multiparticle productions in pp, $\bar{p}p$, ep, e^+e^-, and NN. We also make comments on comparisons with Koba-Nielsen-Olesen and negative-binomial distributions.

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Multiparticle productions in hadronic interactions or in jet hadronizations are effectively one-dimensional (1D) distributions in the rapidity variable or its equivalence due to the sharp cutoff in transverse momenta; thus we can study them in 1D models. With the two interactions, nearest-neighbor influence and an external agitation, the 1D Ising model provides the simplest model in which multiparticle distributions can be calculated. The reasoning of our using the Ising model for multiparticle production is very similar to that of using the Ising model for lattice gas, except the space now in multiparticle production is rapidity. The Ising model is not a dynamical model that governs all detailed dynamics of multiparticle productions, as it is not for phase transitions; however, it may capture the important underlying laws in the seemingly complicated multiparticle productions as it has for phase transitions. Indeed, we show here that we have learned the salient mathematical structures in $P_{\delta\xi}(n)$ that lead to scaling and universality, as well as found a framework to analyze the data of high-energy multiparticle productions in pp, $\bar{p}p$, ep, e^+e^-, and NN. [$\delta\xi$ in $P_{\delta\xi}(n)$ is the interval of the 1D rapidity variable; its full range will be denoted by $\Delta\xi$.]

As reported in a previous publication [1], we have found that the Ising model naturally gives scaling $F_q(\delta\xi)=\tilde{F}_q[F_2]$, $q=3,4,5,\ldots$, and universality, which states that all dynamical dependences are contained in $F_2(\delta\xi)$, thus $\tilde{F}_q[F_2]$ is a universal function for different reactions at different energies. In our previous paper [1] we had derived explicit expressions for $F_q(\delta\xi)$ and $\tilde{F}_q[F_2]$. Here we perform a different calculation in the 1D Ising model and give explicit expressions for the multiplicity distribution $P_{\delta\xi}(n)$, which provides the most basic information on the multiparticle distributions. Other distributions, like factorial moments $F_q(\delta\xi)$ and their scaling and universality $F_q(\delta\xi)=\tilde{F}_q[F_2]$, can be derived from it. Our $P_{\delta\xi}(n)$ is expressed as a γ^n-weighted cluster l-sum of n particles $(c^l/l!)C_{l-1}^{n-1}$ (C_{l-1}^{n-1} is the binomial); γ and c contain the dynamical parameters of the theory and they are determined by the experimental values of $\langle n\rangle_{\delta\xi}$ and $\langle n^2\rangle_{\delta\xi}$. Interestingly we can also express $P_{\delta\xi}(n)$ as a γ^n-weighted sum of products of a Poisson and a negative binomial; see Eq. (4) and the paragraph after it.

After fixing the two parameters of the model by fitting with the data $\langle n\rangle_{\delta\xi}$ and $\langle n^2\rangle_{\delta\xi}$ at various values of $\delta\xi$, the multiparticle distributions $P_{\delta\xi}(n)$ as functions of n are determined in the model and constitute the predictions of the model. They compare well with the data, better for the region of larger $\delta\xi$; see Figs. 1(a) and 1(b). Interestingly the two dynamical parameters in the model become rather flat in their dependences on $\delta\xi$ for larger intervals of $\delta\xi\geq 3$. This implies that in the larger $\delta\xi$ intervals the

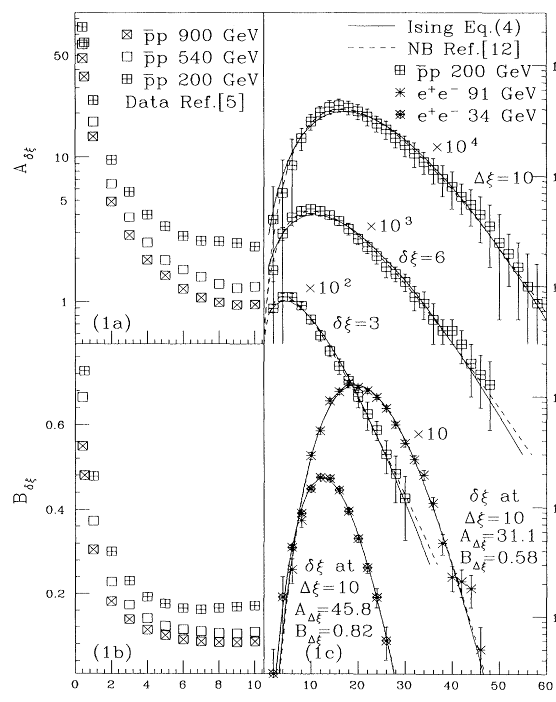

FIG. 1. (a) $A_{\delta\xi}$ and (b) $B_{\delta\xi}$ of Eq. (3) determined from the data $\langle n\rangle_{\delta\xi}$ and $\langle n^2\rangle_{\delta\xi}$ of $\bar{p}p$ [5]. (c) $P_{\delta\xi}(n)$ as a function of n. (Results for $\bar{p}p$ at $\sqrt{s}=540$ and 900 GeV are as good, but not shown.)

Ising model with $\delta\xi$-independent interaction parameters ($A_{\delta\xi}$ and $B_{\delta\xi}$) can also describe the $\delta\xi$ dependences in $P_{\delta\xi}(n)$ and $F_q(\delta\xi)$; thus future data at higher energies, which make larger $\delta\xi$ intervals available, will provide further important information.

From $P_{\delta\xi}(n)$, we can calculate factorial moments,

$$F_q(\delta\xi) \equiv [\langle n(n-1) \cdots (n-q+1)\rangle_{\delta\xi}][\langle n\rangle_{\delta\xi}]^{-q}, \qquad (1)$$

$$q = 1,2,3,\ldots,$$

where $\langle(\,)\rangle_{\delta\xi} \equiv \sum_{n=0}^{\infty}\{(\,)P_{\delta\xi}(n)\}$. These factorial moments F_q of multiparticle productions were pointed out to be important to study in Ref. [2] and were found in Ref. [3] to have the scaling behavior $F_q(\delta\xi) = \tilde{F}_q[F_2]$, and in Ref. [4] to have universality. The data considered include $\bar{p}p$ ($\sqrt{s} = 200$ to 900 GeV) [5] and quark jets from e^+e^- ($\sqrt{s} = 91$ GeV) [6], as well as various nuclear reactions (p, ^{16}O, ^{32}S at 200 GeV/nucleon) [7]. In Ref. [1] we showed that the 1D Ising model can produce these salient features. Here we compare these data with our parameter-free universal distributions $\tilde{F}_q[F_2]$ predicted from the 1D Ising model; see Figs. 2, 3(a), and 3(b). The agreement is excellent in the smaller $F_2(\delta\xi)$ region which comes from larger $\delta\xi$. We anticipate that this trend will continue at higher energies where the available $\delta\xi$ interval will increase.

Our Ising results have no Koba-Nielsen-Olesen (KNO) scaling [8]. Single-negative-binomial distributions (NB) [9] have been popular in characterizing multiparticle distributions, but their dynamical origin is not clear. Ising distributions $P_{\delta\xi}(n)$ are explicitly different functions from single-negative-binomial distributions, though numerically they are similar in comparing with current data; see Fig. 1(c).

It is surprising, yet satisfying, that a simple model like the Ising one captures so many essential features of multiparticle distributions. Later toward the end of the paper we shall describe more specifically the important features to be studied in future higher-energy experiments.

Multiplicity distributions from 1D Ising model.— First we give a brief description of the 1D Ising calculation. The Ising-model Hamiltonian is $H = -\varepsilon\sum_{(i,j)}^{N} S_i \cdot S_j - b\sum_i^N S_i$, where (i,j) means $j = i \pm 1$ and i is summed over all lattice sites N; ε signifies the strength of next-neighbor influence and b the strength of an agitating field. Conventionally S_i represents the spin values $\pm\frac{1}{2}$ at site i. The same as for the lattice gas, defining $n_i \equiv (\frac{1}{2} + S_i) = 1$ or 0, we can interpret n_i as one or no particle production at site i. In this interpretation, b represents the agitation that produces particles and ε the short-range next-neighbor influence among the particles.

The multiplicity distribution in a sublattice N/M is $P_M(n) = (1/Z)\{\{\sum_{\{n_i|\sum n_i = n\}} e^{-\beta H}\}\}$, where $\{n_i|\sum n_i = n\}$ means summing over all possible configurations of n_i in the sublattice N/M with the constraint $\sum n_i = n$. Note that $\sum_{n=0}^{N/M} P_M(n) = 1$ and Z equals the $\sum_{n=0}^{N/M}\{\{\}\}$ part of $P_M(n)$. After some calculations, we obtain

$$P_M(n) = (1/z_M)\left(\gamma^n \sum_{l=1}^{n} [(N/M)C_{l-1}^{(N/M)-n-1} e^{-4\varepsilon\beta l} l^{-1}] C_{l-1}^{n-1}\right) \qquad (2)$$

for $n \neq 0$; and, for $n = 0$, $P_M(0) = 1/z_M$. In Eq. (2), l is

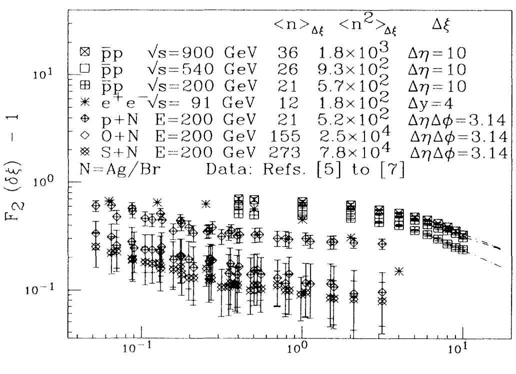

FIG. 2. $F_2(\delta\xi) - 1$ as a function of $\delta\xi$. The dash-dotted lines (straight lines in this plot) are anticipated Ising results for $\delta\xi > 10$ assuming that $A_{\delta\xi}, B_{\delta\xi}$ will stay constant at their present values at $\delta\xi = 10$ as shown in Fig. 1.

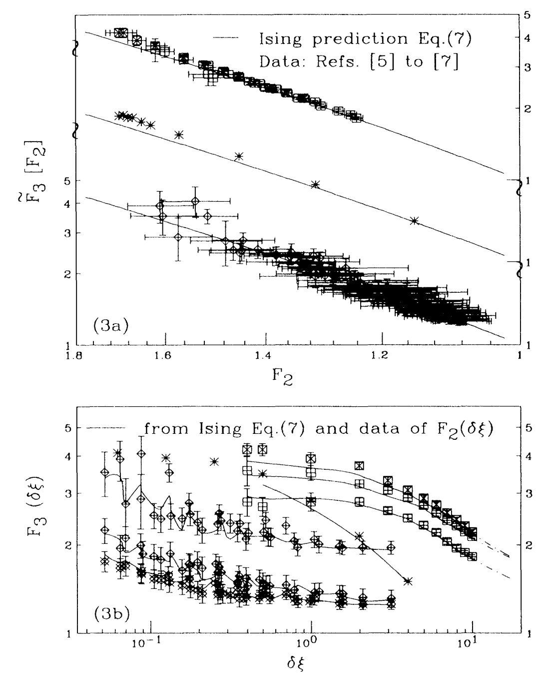

FIG. 3. (a) Universal curve of $\tilde{F}_3[F_2]$ as a function of F_2. (b) $F_3(\delta\xi)$ from the same data as in Fig. 2. The dash-dotted lines are anticipated Ising results as in Fig. 2. (Similarly good results have been obtained for $q = 4$ and 5 for all listed reactions [5–7], but are not shown.)

the number of clusters among n, $\gamma=e^{2\beta b}$, z_M [equal to the $\sum_{n=0}^{N/M}\{\}$ part of Eq. (2)] is the normalization factor, which is related to Z by $z_M=e^{-\beta(\varepsilon-b)N/M}Z$, and $C_k^j=j![k!(j-k)!]^{-1}$ is the binomial. As a result of the γ^n dependence of the $\{\}$ part of $P_M(n)$ in Eq. (2), we can calculate $\langle n\rangle_M$ and $\langle n^2\rangle_M$ by $\partial\gamma$-differentiating the explicitly given z_M: $\langle n\rangle_M=(1/z_M)\,\partial z_M/\partial\gamma$ and $\langle n^2\rangle_M=(1/z_M)\times\partial^2 z_M/(\partial\gamma)^2+(1/z_M)\partial z_M/\partial\gamma$.

To make the connection with multiparticle productions, we take the continuum limit $N\to\infty$ with fixed $\langle n\rangle_{\delta\xi}$ and $\langle n^2\rangle_{\delta\xi}$, which results in two finite parameters $A_{\delta\xi}=Ne^{(-4\beta\varepsilon)}$ and $B_{\delta\xi}=-\beta b$ (notice that $a<1$ and all $a^{N/M}\to 0$) [1]. Further we interpret [10] $M=\Delta\xi/\delta\xi$, $\Delta\xi$ being the whole range of rapidity available, and obtain

$$\langle n\rangle_{\delta\xi}=A_{\delta\xi}[2\sinh(B_{\delta\xi})]^{-2}(\delta\xi/\Delta\xi)\,,$$
$$\langle n^2\rangle_{\delta\xi}=\langle n\rangle_{\delta\xi}^2+\langle n\rangle_{\delta\xi}\coth(B_{\delta\xi})\,, \tag{3}$$

and

$$P_{\delta\xi}(n)=(1/z_p)\{\gamma^n\sum_{l=1}^{n}[c^l/l!]C_{l-1}^{n-1}\}\,,$$
$$P_{\delta\xi}(0)=1/z_p\,, \tag{4}$$

$$z_p=\exp[\gamma c/(1-\gamma)]\,, \tag{5}$$

where $z_p=z_M$ at $N\to\infty$ and z_p is the $\sum_{n=0}^{\infty}\{\}$ part of Eq. (4); using Eq. (3), γ and c can be expressed in terms of $\langle n\rangle_{\delta\xi}$ and $\langle n^2\rangle_{\delta\xi}$ through $\gamma=\exp(-2B_{\delta\xi})$ and $c=A_{\delta\xi}(\delta\xi/\Delta\xi)$.

Notice that $P_{\delta\xi}(n)$ is expressed as a γ^n-weighted cluster l-sum of n-particle productions, which can also be reexpressed as a product of a Poisson distribution in cluster number l, $\{[\gamma c/(1-\gamma)]^l/l!\exp[-\gamma c/(1-\gamma)]\}$, and a negative binomial in $(n-l)$, $\{(1-\gamma)^l\gamma^{n-l}C_{l-1}^{n-1}\}$. Interestingly such distributions are also those given by the Poisson-distributed cluster models [11].

Next we determine from the data $\langle n\rangle_{\delta\xi}$ and $\langle n^2\rangle_{\delta\xi}$, and thus $A_{\delta\xi}$ and $B_{\delta\xi}$ in Eq. (3). Figures 1(a) and 1(b) show the values of $A_{\delta\xi}$ and $B_{\delta\xi}$ at various values of $\delta\xi$ obtained from $\bar{p}p$ data [5]. For e^+e^- and NN reactions we cannot make such a detailed analysis since only $\langle n\rangle_{\Delta\xi}$ and $\langle n^2\rangle_{\Delta\xi}$ are available, shown in Fig. 2; thus we can only obtain $A_{\Delta\xi}$ and $B_{\Delta\xi}$, shown in Fig. 1(c) for e^+e^-. Fortunately, for analyzing universality $\tilde{F}_q[F_2]$, we only need data of $F_q(\delta\xi)$ and $F_2(\delta\xi)$, not those of $\langle n\rangle_{\delta\xi}$ and $\langle n^2\rangle_{\delta\xi}$.

Predictions, comparisons, and outlook.—Once the two parameters $A_{\delta\xi}$ and $B_{\delta\xi}$ are fixed via $\langle n\rangle_{\delta\xi}$ and $\langle n^2\rangle_{\delta\xi}$, Eqs. (4) and (5) specify completely the multiparticle distributions from the 1D Ising model; thus they constitute predictions of the model. The n distributions of $P_{\delta\xi}(n)$ given in Eq. (4) for various values of $\delta\xi$ are shown in Fig. 1(c) together with data from $\bar{p}p$ reactions at $\sqrt{s}=200$ GeV and $\delta\xi=3$, 6, and 10 (at $\sqrt{s}=540$ and 900 GeV, the results are similar but not shown). The results are good, and better for larger $\delta\xi$. Also shown in Fig. 1(c) are the n distributions of $P_{\delta\xi}(n)$ at $\delta\xi=\Delta\xi$ together with determined $A_{\Delta\xi}$ and $B_{\Delta\xi}$ for e^+e^- reactions at $\sqrt{s}=91$ GeV, $\Delta\xi=10$ and at $\sqrt{s}=34$ GeV, $\Delta\xi=10$.

It is interesting to note in Figs. 1(a) and 1(b) that $A_{\delta\xi}$ and $B_{\delta\xi}$ determined from $\bar{p}p$ data flatten out in the larger $\delta\xi$ region, $\delta\xi\geq 3$, which means that for $\delta\xi\geq 3$ the Ising model with $\delta\xi$-independent coupling constants actually can correctly describe the $\delta\xi$ distribution from its phase-space factor $M=\Delta\xi/\delta\xi$ alone. This feature is important to be checked out in future higher-energy data where larger $\delta\xi$ intervals will become available.

We note from Eq. (4) that the Ising $P_{\Delta\xi}(n)$ has no KNO scaling [8], i.e., $\langle n\rangle_{\Delta\xi}P_{\Delta\xi}(n)$ is not a scaling function of $n/\langle n\rangle_{\Delta\xi}$ though approximate KNO scaling can be true for some limited energy range. Also clear is that the functional forms of our Ising $P_{\delta\xi}(n)$ in Eqs. (4) and (5) are different from those of a single negative binomial [12], though a numerical comparison at the present energies shows they are quite similar [see solid and dashed lines in Fig. 1(c)].

From the $P_{\delta\xi}(n)$'s of Eq. (4) we can calculate $F_q(\delta\xi)$. As a result of the γ^n dependence of the $\{\}$ part in $P_{\delta\xi}(n)$ and the nice expression of z_p from Ising, Eq. (5), we obtain the following neat result:

$$F_q(\delta\xi)=\left[\frac{\gamma^q}{z_p}\frac{\partial^q z_p}{(\partial\gamma)^q}\right]\left\{\frac{\gamma}{z_p}\frac{\partial z_p}{\partial\gamma}\right\}^{-q}, \tag{6}$$

which in turn gives

$$\tilde{F}_q[F_2]=1+\sum_{l=1}^{q-1}\frac{(q-1)!q!}{l!(q-l-1)!(q-l)!2^l}[F_2(\delta\xi)-1]^l\,, \tag{7a}$$

where

$$[F_2(\delta\xi)-1]=2\frac{1-e^{-2B_{\delta\xi}}}{A_{\delta\xi}}\frac{\Delta\xi}{\delta\xi}\,. \tag{7b}$$

Equations (7) indicate that all the dependences on the three parameters of the model, $A_{\delta\xi}$, $B_{\delta\xi}$, and $\delta\xi/\Delta\xi$, are absorbed in F_2; thus $\tilde{F}_q[F_2]$, $q=3,4,5,\ldots$, are universal functions, depending solely on F_2.

Since $\langle n(n-1)\cdots(n-q+1)\rangle_{\delta\xi}$ is also the fully integrated q-particle inclusive cross section in the interval $\delta\xi$, the universality $\tilde{F}_q[F_2]$ implies that all fully integrated multiparticle inclusive cross sections are functions of fully integrated two-particle inclusive cross section in the interval $\delta\xi$. This is an essential feature captured by the next-neighbor and agitation interactions in the Ising model.

The scaling and universality of Eqs. (6) and (7) were obtained by us in Ref. [1] from a different calculation, bypassing the calculation of $P_{\delta\xi}(n)$. In the present calculation, we can identify the characteristics in $P_{\delta\xi}(n)$ that lead to scaling and universality in F_q: γ^n dependence in $P_{\delta\xi}(n)$, and $\partial z_p/\partial\gamma=gz_p$, where g has the property that $[g^{-3}\partial^2 g/(\partial\gamma)^2]$ is a function of $[g^{-2}\partial g/\partial\gamma]$. The Ising model provides such an explicit $g=c/(1-\gamma)^2$ and $[g^{-3}\times\partial^2 g/(\partial\gamma)^2]=\frac{3}{2}[g^{-2}\partial g/\partial\gamma]^2$. The single-negative-binomial distribution for $P_{\delta\xi}(n)$ has similar properties which we discuss in Ref. [12], except its g satisfies an equation with $\frac{3}{2}$ on the right-hand side replaced by 2.

The universal curve of $\tilde{F}_q[F_2]$ given in Eq. (7) for $q=3$ is plotted as solid lines in Fig. 3(a). The agreement with the data is surprisingly good [excellent for the smaller $F_2(\delta\xi)$ region], considering they are parameter-free predictions. (Similarly good results have been obtained for $q=4$ and 5 for all the listed reactions [5–7] but are not shown.)

From the data points of $F_2(\delta\xi)$ in Fig. 2 we can also obtain $F_q(\delta\xi)$, $q=3,4,5,\ldots$. They are shown as solid lines in Fig. 3(b) for $q=3$ and compared to data from Refs. [5–7]. (Similar results have been obtained for $q=4$ and 5 for all listed reactions [5–7], but are not shown). Those given by single negative binomials are almost exactly the same as ours for $F_2 \to 1$, and better for $F_2 > 1.5$.

Finally we make a few concluding remarks. We have identified the salient features in $P_{\delta\xi}(n)$ that lead to scaling and universality and have provided a framework to analyze data of high-energy multiparticle productions in pp, $\bar{p}p$, ep, e^+e^-, and NN. The important future tests are in the larger $\delta\xi$ regions. If the trend of constancy of $A_{\delta\xi}$ and $B_{\delta\xi}$ continues as the available $\delta\xi$ interval increases at higher energies, the $\delta\xi$ dependences in $\langle n\rangle_{\delta\xi}$, $\langle n^2\rangle_{\delta\xi}$, and $F_q(\delta\xi)$ can all be anticipated: from Eq. (3), $\langle n\rangle_{\delta\xi} \sim \delta\xi$, $\langle n^2\rangle_{\delta\xi} \sim (\delta\xi)^2$; from Eq. (7), $F_2(\delta\xi)-1 \sim (\delta\xi)^{-1}$ and all $F_q(\delta\xi) \to 1$. They are shown as dash-dotted lines in Figs. 2 and 3(b) for the $\bar{p}p$ reaction. Notice that they are straight lines in Fig. 2 showing $\ln[F_2(\delta\xi)-1]$ vs $\ln(\delta\xi)$. As F_2 decreases for larger $\delta\xi$, there will also be more range available in F_2 to check the universality $\tilde{F}_q[F_2]$. We hope that soon $\langle n\rangle_{\delta\xi}$, $\langle n^2\rangle_{\delta\xi}$, and $P_{\delta\xi}(n)$ will also be provided from other reactions besides $\bar{p}p$, so a full analysis as shown in Fig. 1 can also be done. It will be interesting to see if the gluon jets will have the same universal $\tilde{F}_q[F_2]$ as the quark jets in high-energy nuclear reactions. We need measurements of F_q in the 1D variable $\delta\eta$ in high-energy nuclear reactions, rather than the current available data in $\delta\eta\,\delta\phi$ [7]. These are important measurements for future relativistic heavy-ion experiments.

Indeed, we look forward to more higher-energy multiparticle-production data in different reactions to be further studied and compared with these salient features learned from the 1D Ising model.

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[10] Note that for $A_{\delta\xi}$ and $B_{\delta\xi}$ constant in $\delta\xi$, especially in the small $\delta\xi$ regions, the Ising results do give intermittent behavior $F_q(\delta\xi) \to (1/\delta\xi)^{q-1}$ as $\delta\xi \to 0$; see Eq. (7). In Ref. [1] we tried to obtain the $\delta\xi$ dependence by interpreting $M=(\Delta\xi/\delta\xi)^{1/\alpha}$ and fitted data to find the value of $\alpha \sim 5$ to flatten the $\delta\xi$-dependence of $F_q(\delta\xi)$. Here we have changed the strategy. Rather than presuppose that all the $\delta\xi$ dependence is in M, we leave it open and find out from the data what are the dependences on $\delta\xi$ in the two parameters of the model, $A_{\delta\xi}$ and $B_{\delta\xi}$, when we take the most natural interpretation $M=\Delta\xi/\delta\xi$. In doing so the $\alpha \neq 1$ effect is reflected in $A_{\delta\xi}$ and $B_{\delta\xi}$ having rather strong dependences on $\delta\xi$ in the small $\delta\xi$ region [see Figs. 1(a) and 1(b)]. In any case, the existence of intermittency in multiparticle particle production is hard to establish unambiguously due to the following intrinsic difficulties: The error in the data increases as $\delta\xi$ decreases; as $\delta\xi$ decreases to a value such that the average number of particles produced in that width of $\delta\xi$ becomes comparable to or smaller than q, i.e., $\langle n\rangle_{\Delta\xi}(\delta\xi/\Delta\xi) \lesssim q$, the factorial moment $F_q(\delta\xi)$ is no longer a good quantity to measure fractal behavior. (This is the same situation as measuring a coast line with a ruler smaller than the wavelength of the waves at the shore.) Such difficulties will not be overcome even if the energies of reactions increase since that will only increase the range of rapidity but without improving the errors in small-$\delta\xi$ measurements of F_q or increasing the number of particles in given $\delta\xi$. That is why there are many different opinions on intermittence studies [1]; thus, here, we focus on and emphasize the larger $\delta\xi$ region.

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[12] The negative binomial distribution [9] is given by

$$P_{\delta\xi}^{\rm NB}(n)=(1/z_p^{\rm NB})\{(\gamma^{\rm NB})^n[(n+k_{\delta\xi}-1)!][n!(k_{\delta\xi}-1)!]^{-1}\},$$

where $(k_{\delta\xi})^{-1}=\langle n^2\rangle_{\delta\xi}/\langle n\rangle_{\delta\xi}^2-1/\langle n\rangle_{\delta\xi}-1$. Following similar discussions as for the Ising case and identifying $\gamma^{\rm NB}=\langle n\rangle_{\delta\xi}/(k_{\delta\xi}+\langle n\rangle_{\delta\xi})$ and $z_p^{\rm NB}=(1-\gamma^{\rm NB})^{-k_{\delta\xi}}$, we can easily derive the factorial moments $F_q^{\rm NB}(\delta\xi)$ using an equation exactly like Eq. (6) and obtain the explicit expressions for universality $\tilde{F}_q^{\rm NB}[F_2^{\rm NB}]$.