Kinetic Equations for Long-Wavelength Excitations of the Quark-Gluon Plasma

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We show that long-wavelength excitations of the quark-gluon plasma are described by simple kinetic equations which represent the exact equations of motion at leading order in g. Properties of the so-called "hard thermal loops," i.e., the dominant contributions to amplitudes with soft external lines, find a natural explanation in this approach. In particular, their generating functional appears here as the effective action describing long-wavelength excitations of the plasma.

Significant progress has been achieved recently in understanding long-wavelength excitations of a quark-gluon plasma [1-3]. In equilibrium, at high temperature, such a plasma may be viewed as a gas of weakly interacting, massless quarks and gluons. When coupled to weak and slowly varying perturbations, this system may acquire a collective behavior on a length scale $\sim 1/gT$, where T is the temperature and g the coupling constant, assumed to be small. In this Letter, we present a consistent and physically intuitive description of such long-wavelength phenomena, based on a set of coupled mean-field and kinetic equations, to which the exact equations of motion reduce in leading order in g [4,5].

The kinetic equations encompass all the so-called "hard thermal loops" and clarify the nature of their remarkable properties, left largely unexplained by their original derivation in terms of Feynman diagrams [2,6,7]. These hard thermal loops (HTL) are the dominant corrections, at high temperature, to amplitudes involving soft external lines. (Following the usual terminology, we call an energy or a momentum "soft" when it is of order gT, and "hard" when it is of order T; at equilibrium, most particles are hard.) As they are of the same order of magnitude as the corresponding nonvanishing tree level amplitudes, HTL need to be resummed in higher order calculations [1,2].

Our equations isolate consistently the dominant terms in g in the hierarchy of equations which describe the response of the plasma to weak and slowly varying disturbances, i.e., varying on a scale of order $1/gT$. In doing so, we treat bosons and fermions on the same footing and introduce an average fermionic field $\psi(X)$ in parallel with the average gauge field $A_\mu(X)$. A noteworthy feature of the present approach is that g, besides measuring the interaction strength, controls the wavelength of the soft space-time variations and, for consistency, also the strength of the mean fields. One finds for example that, in order for the deviations away from equilibrium to stay small, the gauge field strength tensor should be at most of order gT^2; then $gA \sim gT$ is of the same order as the derivative of a "slowly varying" quantity, and, when needed, can be combined with it to form a covariant derivative [4,5].

The dominant interactions which determine the re-

sponse of the plasma are those which take place between the hard particles and the soft mean fields. In leading order, we can neglect the direct interaction between the hard particles, which allows us to truncate the equations at the level of the two-point functions. The motion of a given hard particle is only slightly perturbed by its interaction with a soft mean field. However, because the mean fields vary over distances much larger than the interparticle distance ($\sim 1/T$), they affect coherently many hard particles, giving rise to collective "polarization" phenomena. These show up as "induced sources" which add to the external ones in determining the properties of the mean fields.

The equations for the quark and gauge average fields are

$$i\slashed{D}_X \psi(X) = \eta(X) + \eta^{\text{ind}}(X) , \tag{1}$$

and

$$[D_X^\nu, F_{\nu\mu}(X)]^a - g\bar{\psi}(X)\gamma_\mu t^a \psi(X) = j_\mu^a(X) + j_\mu^{\text{ind}\,a}(X) . \tag{2}$$

Here η and j_μ^a are external sources, $a = 1, \ldots, N^2 - 1$ are color indices for the adjoint representation of the SU(N) gauge group, while $\mu, \nu = 0, \ldots, d-1$ are space-time indices ($d = 4$ throughout this work). The covariant derivative is $D_\mu \equiv \partial_\mu + ig t^a A_\mu^a$ and $F_{\mu\nu} \equiv [D_\mu, D_\nu]/(ig)$. When using a covariant gauge, one should also consider equations for ghost mean fields. These are not written here as it turns out that they are trivial, i.e., there are no induced sources for the ghost mean fields [5]. The induced sources can be considered as functionals of the average fields themselves. After functional differentiation, they yield the one-particle irreducible amplitudes with soft external lines. Thus for example the fermion self-energy is given by $\Sigma(x,y) = \delta\eta^{\text{ind}}(x)/\delta\psi(y)$, the gluon polarization tensor by $\Pi_{\mu\nu}^{ab}(x,y) = \delta j_\mu^{\text{ind}\,a}(x)/\delta A_b^\nu(y)$, etc. One outcome of our work is a set of consistent expressions for the induced sources, at leading order in g. As we shall see, these represent generating functionals for all the hard thermal loops.

In leading order, the induced sources can be expressed entirely in terms of two-point functions. For example, $\eta^{\text{ind}}(x) = g\gamma^\nu t_a \langle A_\nu^a(x)\psi(x)\rangle_c$, where the subscript c indi-

cates a connected expectation value which, in leading order, involves only the hard particles. The abnormal quark-gluon propagator which enters η^{ind} is nonvanishing only in the presence of the fermionic mean field ψ. It vanishes in equilibrium, as does the induced color current j^{ind}. In leading order, it is possible to write this current as the sum of two terms, $j^{\text{ind}} \equiv j^A + j^\psi$, where j^A is the current induced by a gauge field, while j^ψ is that induced by a fermionic mean field. Note, however, that the effects of the two kinds of fields are not totally independent: j^ψ depends on A, as required by gauge covariance. Furthermore, both components receive contributions from various species of hard particles and it is convenient to calculate separately the contributions of fermionic and bosonic particles. Accordingly, we write $j^A = j_f^A + j_b^A$ and $j^\psi = j_f^\psi + j_b^\psi$. The fermion pieces involve only the hard quark propagator in the presence of the fields: $j_{f\mu}^A + j_{f\mu}^\psi = gt^a \langle \bar{\psi}(x) \gamma_\mu t^a \psi(x) \rangle_c$. In a covariant gauge, the bosonic pieces involve hard gluons and ghosts [5].

In order to implement the condition that the average fields are slowly varying, it is convenient to use the Wigner transform of the two-point functions, such as

$$\mathscr{S}(k,X) \equiv \int d^4s\, e^{ik\cdot s} \langle \bar{\psi}(X-s/2)\psi(X+s/2)\rangle_c , \qquad (3)$$

for the quark propagator. Note that, in contrast to other authors, we do not insist on defining manifestly gauge covariant Wigner functions [8]. Covariance will be recovered later, when calculating physical quantities. The induced sources are then integrals over k of Wigner functions, which we refer to as k-space densities. For example,

$$\eta^{\text{ind}}(X) = g \int \frac{d^4k}{(2\pi)^4} t_a \gamma^\nu \mathscr{H}_\nu^a(k,X) , \qquad (4)$$

where $\mathscr{H}_\nu^a(k,X)$ is the Wigner transform of the quark-gluon propagator $K_\nu^a(x,y) \equiv \langle A_\nu^a(y)\psi(x)\rangle_c$. Thus $\mathscr{H}(k, X) \equiv t^a \mathscr{H}^a(k,X) \equiv t^a \gamma^\nu \mathscr{H}_\nu^a(k,X)$ is the k-space density for $\eta^{\text{ind}}(X)$. The densities for the induced currents are denoted by $\mathscr{J}_{f,b}^A(k,X)$ and $\mathscr{J}_{f,b}^\psi(k,X)$. For example, the fermionic density is $\mathscr{J}_{f\mu}^a(k,X) = \text{Tr}\gamma_\mu t^a \mathscr{S}(k,X)$, where the trace refers to both spinor and color indices.

At leading order in the coupling g, the Schwinger-Dyson equations for two-point functions reduce to the following equations for the densities of the induced sources (technical details on the derivation of these equations will be given in [5]):

$$(k \cdot D_X)\mathscr{H}(k,X) = -i\frac{g}{2}(d-2)C_f[\Delta(k)+\bar{\Delta}(k)]\mathscr{K}\psi(X) , \qquad (5)$$

$$[k \cdot D_X, \mathscr{J}_{f\mu}^A(k,X)]^a = 2gN_f k_\mu k^\rho F_{\rho\nu}^a \partial_k^\nu \bar{\Delta}(k) , \qquad (6)$$

$$[k \cdot D_X, \mathscr{J}_{f\mu}^\psi(k,X)]^a = igk_\mu\{\bar{\psi}(X)t^a\mathscr{K}(k,X) - \bar{\mathscr{H}}(k,X)t^a\psi(X)\} - gk_\mu f^{abc}\{\bar{\psi}(X)t^b\mathscr{K}^c(k,X) - \bar{\mathscr{H}}^b(k,X)t^c\psi(X)\} , \qquad (7)$$

$$[k \cdot D_X, \mathscr{J}_{b\mu}^\psi(k,X)]^a = gk_\mu f^{abc}\{\bar{\psi}(X)t^b\mathscr{K}^c(k,X) - \bar{\mathscr{H}}^b(k,X)t^c\psi(X)\} , \qquad (8)$$

$$[k \cdot D_X, \mathscr{J}_{b\mu}^A(k,X)]^a = gN(d-2)k_\mu k^\rho F_{\rho\nu}^a \partial_k^\nu \Delta(k) . \qquad (9)$$

In these equations, $\mathscr{H}(k,X) = \mathscr{K}^\dagger(k,X)\gamma^0$, $C_f \equiv (N^2-1)/2N$ is the quark Casimir, N_f is the number of quark flavors, and f^{abc} denote the structure constants of SU(N). Furthermore, $\Delta(k) \equiv \rho_0(k)N(k_0)$ and $\bar{\Delta}(k) \equiv \rho_0(k)n(k_0)$, where $\rho_0(k) = 2\pi\epsilon(k_0)\delta(k^2)$ is the spectral function for free massless particles, and $N(k_0)$ and $n(k_0)$ denote respectively boson and fermion occupation factors. The factor $d-2$ in Eqs. (5) and (9) reflects the fact that only the transverse gluons effectively contribute to the densities.

Equations (5)–(9) have a number of interesting properties: (i) They are independent of the gauge fixing parameter λ which enters calculations in general covariant gauges [4,5]. (ii) On their right-hand sides, all possible vacuum contributions cancel. (iii) They transform covariantly under a local gauge transformation of the mean fields A_μ, ψ, and $\bar{\psi}$. The densities $\mathscr{J}^\psi$, $\mathscr{J}^\psi$, and $\mathscr{H}$ involve Wigner transforms which are gauge covariant in leading order. The current induced by a gauge field involves k-space densities, $\mathscr{J}_f^A$ and $\mathscr{J}_b^A$, which are derived from non-covariant Wigner functions. However, these densities are defined up to a total derivative with respect to k which does not contribute to the integrated current. We have used this freedom in order to make the densities explicitly covariant [4,5]. (iv) The symmetry between Eqs. (6) and

(9) reflects the fact that hard quarks and gluons respond similarly to a soft gauge field. The same symmetry is apparent in Eq. (5) expressing the effect of the fermionic mean field on the hard particles. (v) The equations contain the covariant line derivative $k \cdot D_X$ on their left-hand sides, and a factor $\delta(k^2)$ on their right-hand sides. This reflects the elementary dynamics of the hard particles: They undergo essentially forward scattering on the mean fields, and remain on their unperturbed mass shell.

As these remarks strongly suggest, the motion of the hard particles described by Eqs. (5)–(9) exhibits many features of classical dynamics. This becomes more transparent if one makes explicit the structure of the various densities implied by these equations. For instance, Eq. (6) implies

$$\mathscr{J}_{f\mu}^A(k,X) = 2k_\mu N_f t^a[2\pi\delta(k^2)]$$

$$\times [\theta(k^0)\delta n_+^a(\mathbf{k},X) + \theta(-k^0)\delta n_-^a(-\mathbf{k},X)] , \qquad (10)$$

where $\delta n_\pm = \delta n_\pm^a t^a$ are fluctuations in the quark color densities induced by the gauge field. These fluctuations satisfy (with $\epsilon_k \equiv |\mathbf{k}|$)

$$[v \cdot D_X, \delta n_{\pm}(\mathbf{k}, X)]^a = \mp g \mathbf{v} \cdot \mathbf{E}^a(X) \frac{dn(\epsilon_k)}{d\epsilon_k} . \qquad (11)$$

In the Abelian case, this equation coincides with the linearized Vlasov equation. Here, the color electric field not only modifies the motion of the particle, but also induces a "precession" of the densities in color space. The other equations may be given similar interpretation. Thus, Eq. (5) for the quark-gluon Wigner function $\mathcal{K}$ describes fluctuations where, under the action of a soft fermionic mean field, quarks are converted into gluons and vice versa.

The total current induced by a gauge field A is $j^A = j_f^A + j_b^A$. Its k-space density, $\mathcal{J}^A \equiv \mathcal{J}_f^A + \mathcal{J}_b^A$, satisfies

$$[k \cdot D_X, \mathcal{J}_\mu^A(k, X)] = g k_\mu k \cdot F(X)$$
$$\cdot \partial_k [2N_f \tilde{\Delta}(k) + N(d-2)\Delta(k)] . \quad (12)$$

This equation generalizes the Vlasov equation to non-Abelian plasmas. Previous attempts to derive such an equation led to more intricate results. However, they were based on different approximation schemes which mix leading and nonleading contributions in g and, as such, are not entirely consistent [8]. One can also combine Eqs. (7) and (8) into a single equation for the total current density induced by the fermionic fields, $\mathcal{J}^\psi \equiv \mathcal{J}_f^\psi + \mathcal{J}_b^\psi$,

$$[k \cdot D_X, \mathcal{J}_\mu^\psi(k, X)]$$
$$= i g k_\mu t^a \{\bar{\psi}(X) t^a \mathcal{K}(k, X) - \mathcal{K}(k, X) t^a \psi(X)\} . \quad (13)$$

This equation is similar to the corresponding one in the Abelian case [4]. In doing the sum of Eqs. (7) and (8), the typical non-Abelian effects cancel; these are con-

tained for example in the second braces in Eq. (7), and involve the three-gluon vertex leading to gauge field insertions on the hard gluon lines. This kind of cancellation was first noted by Taylor and Wong [9] in relation with the HTL's for amplitudes involving one pair of quarks and any number of soft gluons (albeit their proof is only explicit up to three external gluons).

We have thus reduced the set of equations (5)–(9) to three fundamental equations, namely, Eqs. (5), (12), and (13) for the densities of the induced sources. From the gauge-covariant character of these equations, it follows that, under local gauge transformations, η^{ind} transforms like ψ, while j^{ind} transforms like $F_{\mu\nu}$. Provided the external sources are chosen so as to satisfy the same property, the mean-field equations (1) and (2) are then gauge covariant, as are the classical equations of motion derived from the QCD action.

Equations (5), (12), and (13) contain all the information on the generalized polarizability of the plasma. After solving Eq. (5), with retarded boundary conditions, we compute the fermionic induced source according to Eq. (4) and obtain

$$\eta^{\text{ind}}(X) = -i\omega_0^2 \int \frac{d\Omega}{4\pi} \psi \int_0^\infty du\, U(X, X - vu)\psi(X - vu)$$
$$\equiv \int d^4Y\, \delta\Sigma_A(X, Y)\psi(Y) , \qquad (14)$$

with $\omega_0^2 \equiv C_f(g^2 T^2/8)$ and $v^\mu \equiv (1, \mathbf{v})$, $\mathbf{v} \equiv \mathbf{k}/\epsilon_k$. The angular integral runs over all directions of $\mathbf{v}$, and $U(x, y)$ is the parallel transporter along a straight line joining x and y [4]. The kernel $\delta\Sigma_A$ is the self-energy of a soft fermion propagating in a background gauge field and it may be easily obtained from this equation. The current induced by fermionic mean fields results from Eq. (13):

$$j_\mu^\psi(X) = g\omega_0^2 t^a \int \frac{d\Omega}{4\pi} v_\mu v_\nu \int_0^\infty dt \int_0^\infty ds\, \bar{\psi}(X - vt)\gamma^\nu U(X - vt, X) t^a U(X, X - vs)\psi(X - vs)$$

$$\equiv g t^a \int d^4Y_1 d^4Y_2\, \bar{\psi}(Y_1)\delta\Gamma_{\mu,a}^A(X; Y_1, Y_2)\psi(Y_2) . \qquad (15)$$

The correction $\delta\Gamma^A$ to the quark-gluon vertex may be easily read out from this equation. Finally, the current induced by soft gauge fields is determined from Eq. (12) to be

$$j_\mu^A(X) = 3\omega_p^2 \int \frac{d\Omega}{4\pi} v_\mu \int_0^\infty du\, U(X, X - vu)F_{0j}(X - vu)$$
$$\times v^j U(X - vu, X) . \qquad (16)$$

Here $\omega_p^2 \equiv (g^2 T^2/9)(N + N_f/2)$ is the plasma frequency. By successive functional differentiation with respect to A of Eq. (16) we derive corrections to the equilibrium amplitudes for soft gluon fields. All the amplitudes obtained in this way, as well as those contained in $\delta\Sigma_A$ and in $\delta\Gamma_\mu^A$, coincide with the HTL's of the diagrammatic approach.

Equations (14)–(16) imply the following covariant conservation laws for the induced currents [5]:

$$[D^\mu, j_\mu^A] = 0 \qquad (17)$$

and

$$[D^\mu, j_\mu^\psi] = i g t^a (\bar{\psi} t^a \eta^{\text{ind}} - \bar{\eta}^{\text{ind}} t^a \psi) . \qquad (18)$$

By differentiating these equations with respect to the fields, one obtains relations between HTL referred to as "QED-like Ward identities" in Refs. [2,7]. By using Eq. (18), together with the equation of motion for ψ, we can verify that the total current associated to fermionic fields, $g\bar{\psi}\gamma_\mu t^a \psi + j_\mu^{\psi a}$, is conserved in the absence of external sources.

By eliminating the induced sources from Eqs. (1),(2), using their explicit expressions (14)–(16), one obtains nonlinear equations of motion which generalize the Maxwell equations in a polarizable medium. In particular, for vanishing external sources, these equations describe the normal modes of the plasma. Note that in general, as a

consequence of gauge covariance, quark and gluon modes mix.

These nonlinear equations for the mean fields can be generated by the minimal action principle applied to an effective action $S_{\text{eff}} = S_0 + S_{\text{ind}}$. Here S_0 is the classical QCD action, while S_{ind} contains the effects of the interactions between the soft fields and the hard particles of the plasma. It follows that S_{ind} must satisfy $\delta S_{\text{ind}}/\delta\bar{\psi}(X) = \eta^{\text{ind}}(X)$ and $\delta S_{\text{ind}}/\delta A_a^\mu(X) = j_\mu^{\text{ind}\, a}(X)$. These conditions are satisfied formally, i.e., on the space of fields for which $v \cdot D_x$ never vanishes [5,9], by $S_{\text{ind}} = S_f + S_b$, with

$$S_f = \omega_0^2 \int \frac{d\Omega}{4\pi} \int d^4X \int d^4Y \, \bar{\psi}(X)\langle X| \frac{v}{i(v\cdot D)}|Y\rangle \psi(Y) \tag{19}$$

and

$$S_b = \frac{3}{2}\omega_p^2 \int \frac{d\Omega}{4\pi} \int d^4X \int d^4Y \, \text{tr}\left[F_{\mu\lambda}(X)\langle X| \frac{v^\mu v_\nu}{(v\cdot\tilde{D})^2}|Y\rangle F^{\nu\lambda}(Y) \right], \tag{20}$$

where $\tilde{D}$ is the covariant derivative in the adjoint representation ($\tilde{D}F_{\mu\nu} = [D, F_{\mu\nu}]$), and the trace acts on color indices only. This action coincides with the generating functional for HTL's derived in [9] on the basis of gauge invariance. Here S_{eff} has a different, more physical, interpretation: It is the classical action describing long-wavelength excitations in the hot quark-gluon plasma, at leading order in the coupling g.

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