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Nonperturbative (but Approximate) Method for Solving Differential Equations and Finding Limit Cycles

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A nonperturbative method for solving differential equations and for finding limit cycles is proposed and is illustrated on the anharmonic oscillator and on the Van der Pol equation. It is shown to give the amplitude, period, and equation of the limit cycle with a better accuracy than any perturbative results so far obtained.

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The idea that lies behind perturbation theory is to find the quantity of interest as a series expansion in a small parameter [1]. The obvious drawback of this method is that we are often interested in the large parameter regime of the theory under study. To remedy this difficulty, several ideas have been proposed such as improvement of the convergence of the series by resummation (Padé approximants for instance) or finding an "artificial" expansion parameter in order to obtain nonperturbative information in the "physical" parameter ($1/N$ expansion in field theory for instance). I present in this Letter a somewhat different philosophy that I illustrate in the context of nonlinear oscillators. I deal with one of the important problems of this physics, namely, to find the limit cycles of differential equations, i.e., the periodic trajectories of the motion [2]. In this Letter, I begin to apply the method to the determination of anharmonic oscillator trajectories (x^4 and x^6 potentials) far in the nonperturbative regime; then I compute the equation of the limit cycle of the Van der Pol equation [3]. In both cases, I obtain approximate solutions not based on a series expansion. As is explicitly shown, the accuracy thus obtained is almost independent of the value of the parameter used to solve these equations in the usual perturbative expansion. Moreover, it is always possible to achieve a far better accuracy than was obtained previously by perturbation theory. I show this explicitly in the example of the Van der Pol limit cycle, though in this case the parameters of the cycle have been calculated up already to the 163rd order of perturbation [4].

The most interesting features of this method are its extreme simplicity and its very good accuracy in a wide range of parameter values. No mathematical proof is given here and a complete study of the accuracy of the method as well as some of its refinements will be given in a forthcoming publication.

I consider in the following a particular class of differential equations in which I exemplify the method. Let us call $x(t)$ the solution of the second-order differential equation,

$$f(x(t), x'(t), x''(t)) = 0, \tag{1}$$

subject to the initial conditions

$$x(t_0) = x_0, \quad x'(t_0) = x'_0. \tag{2}$$

The principle of the method is to replace Eqs. (1) and (2) by a linear differential equation with an explicit time-dependent right-hand side. In the language of classical mechanics, this means that we look for an external force acting on a particle in a harmonic potential that forces it to follow the trajectory $x(t)$. This force always exists and is given by

$$x''(t) + \omega^2 x = F(t), \tag{3}$$

where ω is still a free parameter. Note that F depends on the particular differential equation we are studying and also on the initial conditions (2). Equations (1) and (2) are completely equivalent to (3) and (2). Now the principle of the method is to make an ansatz F_{ans} for F and to

search, by a variational-like principle, for the $F_{\rm ans}$ that best approximates the true F. There lies the advantage of this formulation: It allows one to make explicit ansatz and once such an ansatz is given, to integrate trivially (1). We restrict ourselves in this Letter to periodic trajectories where this approximate scheme appears to be powerful.

Let us call $x_{\rm ans}(t)$ the solution of (3) with $F_{\rm ans}$,

$$x''_{\rm ans}(t) + \omega^2 x_{\rm ans}(t) = F_{\rm ans}(t), \tag{4}$$

and δx the difference between x and $x_{\rm ans}$,

$$x(t) = x_{\rm ans}(t) + \delta x(t). \tag{5}$$

The equation obeyed by δx follows from (1):

$$f(x_{\rm ans} + \delta x, x'_{\rm ans} + \delta x', x''_{\rm ans} + \delta x'') = 0. \tag{6}$$

If for any t, $x_{\rm ans}$ is close enough to x so that

$$|\delta x| \ll |x_{\rm ans}|; \quad |\delta x'| \ll |x'_{\rm ans}|; \quad |\delta x''| \ll |x''_{\rm ans}|, \tag{7}$$

then

$$f(x_{\rm ans}(t), x'_{\rm ans}(t), x''_{\rm ans}(t)) \sim O(\delta x). \tag{8}$$

In practice, $x_{\rm ans}$ is expanded on a basis of functions, in our case as a Fourier sum (this is equivalent to choosing $F_{\rm ans}$):

$$x_{\rm ans}(t) = \sum_{k=1}^{N} (x_k \sin k\omega t + y_k \cos k\omega t). \tag{9}$$

We now determine our "best" $x_{\rm ans}$ by imposing the self-consistent condition that δx is small compared to $x_{\rm ans}$ and that δx involves only harmonics higher than N. This approximation means that all the contributions to harmonics less than or equal to N are taken into account by $x_{\rm ans}$. This requirement is reminiscent of ordinary perturbation theory. Condition (8) now becomes a set of coupled algebraic equations, one for each Fourier component. Therefore, Eq. (8) determines the approximate solution $x_{\rm ans}(t)$. Then, to go further, we can use Eq. (8) to determine δx at first order, i.e., after having kept only leading terms in it.

Let us see how this extremely simple approach works on specific examples. We first consider the anharmonic oscillator with an x^4 potential:

$$x'' = -\omega_0^2 x - g x^3. \tag{10}$$

The origin of time is chosen such that the initial conditions are

$$x(0) = 0; \quad x'(0) = x'_0. \tag{11}$$

In ordinary perturbation theory, one looks for the solution of (10) as a series in g. Here we replace (10) and (11) by the equation

$$x''_{\rm ans} = -\omega^2 x_{\rm ans} + F_{\rm ans}(g, x'_0), \tag{12}$$

and we choose $x_{\rm ans}$ as in (9),

$$x_{\rm ans}(t) = a \sin \omega t + \sum_{k=1}^{N} x_{2k+1} \sin(2k+1)\omega t, \tag{13}$$

which is consistent with (11). We particularize a since it is convenient to determine x'_0 from it (see Table I). The set of x_{2k+1} and ω are therefore the free parameters. δx is given by (5) and obeys the equation

$$\delta x'' = -\omega_0^2\, \delta x - (\omega_0^2 - \omega^2) x_{\rm ans} - g(x_{\rm ans} + \delta x)^3 - F_{\rm ans}. \tag{14}$$

Note that up to now, no approximation has been made. We now determine $x_{\rm ans}$ by imposing the self-consistent condition that δx is small compared to $x_{\rm ans}$ and that it starts with harmonics higher than $2N+1$. Projecting out the harmonics greater than $2N+1$ in Eq. (14), we obtain

$$(\omega_0^2 - \omega^2) x_{\rm ans} + g x_{\rm ans}^3 + F_{\rm ans} \sim 0. \tag{15}$$

This algebraic equation together with (11) leads to the $N+1$ equations that uniquely determine $\omega, x_3, \ldots, x_{2N+1}$. We clearly see in this example that this approximation scheme does not require a small g. It only requires that $|\delta x|$ be small with the ansatz (13). As I show now it leads to a sequence of approximations, the accuracy of which depends on the ansatz, i.e., on N. We start by taking $N = 1$:

$$x_{\rm ans}(t) = a \sin \omega t + x_3 \sin 3\omega t, \tag{16}$$

where $\omega = \omega(a, g)$, $x_3 = x_3(a, g)$. Equation (15) implies that

$$\begin{aligned} &\omega^2 = \omega_0^2 + \tfrac{3}{4} g \left(a^2 - a x_3 + 2 x_3^2\right), \\ &(\omega^2 - \omega_0^2) x_3 + 8\omega^2 x_3 + \tfrac{1}{4} g a^3 - \tfrac{3}{4} g x_3^3 - \tfrac{3}{2} g a^2 x_3 = 0, \end{aligned} \tag{17}$$

so that x_3 is the solution of a third degree equation. I give in Table I a list of values of $\omega(a, g)$ and $x_3(a, g)$ and I compare ω with the "true" ω, i.e., the one obtained by numerical integration. Let us now show that Eq. (14) from which we discard all δ terms but $\delta x''$ enables deter-

TABLE I. Period and amplitude of the anharmonic oscillator with x^4 potential and $\omega_0 = 1$. The second column refers to the ansatz $2N+1 = 3$, the third to $2N+1 = 7$, and the last one to the exact values.

	T $x_{\max}$	T $x_{\max}$	$T^{\rm ex}$ $x^{\rm ex}_{\max}$
$a = 1$	4.730 04	4.730 002 6	4.730 004 2
$g = 1$	1.019 02	1.019 053 3	1.019 052 6
$a = 100$	0.022 402	0.022 397 0	0.022 397 4
$g = 10$	104.68	104.705	104.709
$a = 100$	0.007 084	0.007 082 6	0.007 082 7
$g = 100$	104.48	104.705	104.71

mining δx. First, since δx is supposed to be small we can neglect all $(\delta x)^2$ and $(\delta x)^3$ terms. On the other hand, the δx term reads $(\omega_0^2 + 3gx_{\rm ans}^2)\delta x$. Since δx starts with the fifth harmonic, $\delta x \sim \delta x_5 \sin 5\omega t$, which is also its leading term, and since ω^2 is always at least of the same order of magnitude as ω_0^2 and $3ga^2$, then $\delta x'' \sim 25\,\omega^2 \delta x_5 \sin 5\omega t$ is large compared to δx that we can therefore also neglect. Equation (14) becomes

$$\delta x'' \simeq -(\omega_0^2 - \omega^2)x_{\rm ans} - gx_{\rm ans}^3 - F_{\rm ans}. \tag{18}$$

Let us remark that if $x_{\rm ans}$ contains harmonics up to $(2N+1)\omega$, $\delta x''$ is of order $(2N+3)^2\omega^2\delta x_{2N+3}\sin(2N+3)\omega t$ and the approximation is therefore better and better as N increases. We also remark from (18) that δx is entirely determined by $x_{\rm ans}$. In our case, $N = 1$, we have

$$\delta x = \delta x_5 \sin 5\omega t + \delta x_7 \sin 7\omega t + \delta x_9 \sin 9\omega t, \tag{19}$$

with

$$\delta x_5 = \frac{3g}{100\omega^2}x_3(x_3 - a); \quad \delta x_7 = -\frac{3g}{196\omega^2}ax_3^2;$$
$$\delta x_9 = -\frac{g}{324\omega^2}x_3^3. \tag{20}$$

The same calculations can be performed with $N = 2, 3 \ldots$. In these cases, δx is a sum of $\delta x_{2k+1}\sin(2k+1)\omega t$ with $k \in [N+1, 3N+1]$. By comparing our solution with the exact solution, it appears clearly that the accuracy of our results is excellent even for g and a large. It is also obvious that the approximate solution converges very fast with N and that the first values of N are enough to obtain accurate results even in the nonperturbative regime where $gx^3 \gg \omega_0^2 x$ at the turning point (which is also the point of maximum amplitude). In any examples I have studied the δx_{2k+1}'s are very small when k is large and for the lowest k, $2k+1 = 2N+3$, $\delta x_{2N+3}(a, g)$ determined at level N by (18) is very close to $x_{2N+3}(a, g)$ determined at level $N+1$ by (15). This justifies *a posteriori* the approximation leading to Eq. (18).

The same study can be done with the x^6 potential with almost the same conclusions. This shows in particular that the accuracy of these results is not specific to the quartic potential. Note also that all the results thus obtained are robust to the choice of ansatz: It is possible to include cosine terms as well as even harmonics in the ansatz (13) without changing the accuracy since the order of magnitude of the coefficients of these terms are all extremely small (about 10^{-14} for the coefficient of $\sin 2\omega t$).

Let us now study a somewhat more sophisticated problem, the determination of the limit cycle of the Van der Pol equation:

$$x'' + g(x^2 - 1)x' + x = 0. \tag{21}$$

This equation is known to have a limit cycle, i.e., a periodic trajectory in phase space which is an attractor of the dynamics whatever g is [2,3]. This equation has been extensively studied and the equation of the limit cycle calculated in perturbation theory up to order g^{163} [4]. I now show that the method outlined previously allows one to determine the period of the limit cycle as well as its parametric equation in phase space. The method is the same as for the anharmonic oscillator and I study only a few numerical examples. A complete study will be published elsewhere. The functions $x_{\rm ans}$ and δx are again expanded as Fourier sums:

$$x_{\rm ans} = \sum_{k=1}^{N}[x_{2k+1}\sin(2k+1)\omega t + y_{2k+1}\cos(2k+1)\omega t],$$
$$\delta x = \sum_{k=N+1}^{3N+1}[\,\delta x_{2k+1}\sin(2k+1)\omega t + \delta y_{2k+1}\cos(2k+1)\omega t]. \tag{22}$$

Here again, inclusion of even harmonics would not change the results since anyway they are found to be extremely small. The projected equation for x_n and y_n reads in this case

$$-g(x_{\rm ans}^2 - 1)x'_{\rm ans} + (\omega^2 - 1)x_{\rm ans} - F_{\rm ans} \sim 0. \tag{23}$$

To determine the limit cycle, we also have to compute the initial conditions since it is only for these special conditions that the trajectory is periodic. However, expanding $x_{\rm ans}$ as a Fourier sum already supposes the periodicity. Therefore, we just have to impose that, for instance, at $t = 0$, the particle is at the point of maximum amplitude with a vanishing velocity. These two conditions together with (23) are enough to determine the x_{2k+1}, y_{2k+1} and ω and therefore the parametric equation in phase space of the limit cycle $(x(t), x'(t))$ as well as its amplitude and period. I show in Table II the numerical values of the period and amplitude obtained with three ansatz for $g = 1$ and compare them with those obtained by a highly accurate numerical integration of the equation. Instead of giving all the x_{2k+1}, y_{2k+1}, which is not very illuminating, I show in Fig. 1 the difference between the "true" $x(t)$ obtained by integration and $x_{\rm ans}(t)$ for $2N+1 = 17$. The accuracy is again excellent (more accurate by at least 5 orders of magnitude for the amplitude and 10 orders

TABLE II. Period and amplitude of the limit cycle of the Van der Pol equation with $g = 1$.

$2N+1=5$	T	6.6634
	$x_{\rm max}$	2.02
$2N+1=17$	T	6.663 286 859 322 8
	$x_{\rm max}$	2.008 62
$2N+1=29$	T	6.663 286 859 323 130 1
	$x_{\rm max}$	2.008 619 860 9
Exact	T	6.663 286 859 323 130 1
	$x_{\rm max}$	2.008 619 860 87

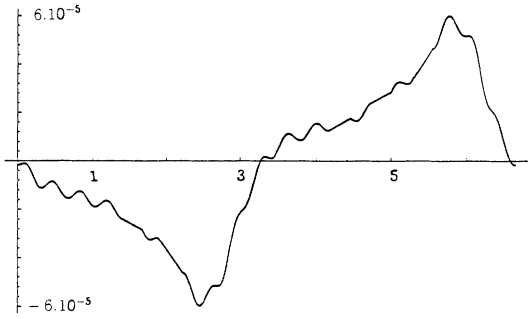

FIG. 1. Plot of the difference between the $x(t)$ obtained by numerical integration and $x_{\text{ans}}(t)$ on one period for the ansatz $2N+1=17$.

of magnitude for the period than that obtained with the usual perturbative expansion at order g^{163} accompanied by a Padé resummation [4]) though the ansatz chosen could seem very crude. This is indeed one of the main advantages of this method, that simple ansatz lead to high accuracy results.

To summarize, we can conclude from the results thus obtained that the method presented here is extremely simple in its principle, quite easy to use, and gives a very good numerical accuracy, even with a simple ansatz. It seems plausible that this method enables one to determine any kind of limit cycle at almost any accuracy. Though I did not prove it, it seems very probable that as higher harmonics are included in the ansatz, $x_{\text{ans}}+\delta x$ converges to the true solution whatever g is. This is a very interesting property of this method compared to the usual perturbative expansion where the series are usually asymptotics and need Padé-Borel resummation to make sense. The only limitation of the present method probably appears when the Fourier spectrum is very wide. Let me, however, mention that in this case no method is known to determine the limit cycles and that nowadays symbolic calculus on computers allows one to handle hundreds of harmonics at a time (for the $2N+1=29$ ansatz, the whole calculation made on a workstation with Mathematica took no more than half an hour). Let me also emphasize that the present method was used in this Letter to determine periodic solutions of differential equations but that it can be applied each time a function can be efficiently expanded on a basis of functions, i.e., that it does not involve too many components. It would actually be very interesting to investigate the possibility of using a different basis such as wavelets, for instance, to find the solutions of differential equations. Let me finally point out that the starting point of this study was a reflection about perturbation theory in quantum field theory and that it would be fascinating if the same type of philosophy coud also be applied in this domain.

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