Dynamical Confinement of Twisted Soliton Pairs in Biaxial Ferromagnets

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The dynamics of a classical continuum model of a ferromagnet with exchange and anisotropy energies of hard- and easy-axis type is considered. Exact solutions of soliton pairs with vanishing center of mass velocity are presented. They reveal the counterintuitive existence of a family of bound states of a repulsive, twisted domain wall (soliton) pair. These breather states have energies which lie above the continuum of the corresponding scattering states of two solitons.

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It is widely believed that a classical bound state only exists if the energy of its constituents exhibits a local minimum. In this Letter, it is shown for the first time that the dynamics of ferromagnetic domain walls provides an apparent counterexample to this presumption.

Ferromagnetic domain walls (solitons) describe the gradual change of the magnetization between two uniform domains of minimal anisotropy energy. Since the exchange interaction can be regarded as a stiffness of the magnetic system, two untwisted domain walls experience an attractive force while two twisted domain walls are repulsive. By means of an exact, time dependent solution of the conservative Landau-Lifshitz equation we shall show the surprising fact that two repulsive solitons can form a bound state. This state is of purely dynamical origin and its existence cannot be predicted by energy or topology considerations. The two solitons oscillate around their center of mass while the magnetization at the domain wall centers is precessing monotonically.

The energy of this breather state is shown to be higher than any scattering state of two solitons. This is in sharp contrast to other nonlinear systems where breathing states [1] of two solitons always have a smaller energy than two static solitons. This surprising effect is intimately connected to the precessional nature of the equations of motion. They are not Lorentz invariant and the kinetic energy of moving solitons is bounded in contrast to relativistic systems. In combination with the integrability of the system, it is ultimately this boundedness of the kinetic energy that gives rise to this peculiar phenomenon. The exchange energy cannot be fully converted into kinetic energy thus leading to the confinement of the two solitons in a high energy state. In this connection, note also the failure of the common construction of momentum in classical spin chains [2].

The ferromagnet is described within a classical continuum model. In dimensionless units the magnetization is represented by the unit vector $\mathbf{M} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ and the energy per unit area of planar symmetric structures is given by

$$\mathcal{E} = \int_{-\infty}^{\infty} dx \left\{ \frac{1}{2}[(\partial_x\theta)^2 + \sin^2\theta(\partial_x\phi)^2] + \frac{Q^{-1}}{2}\sin^2\theta\cos^2\phi + \frac{1}{2}\sin^2\theta \right\}, \tag{1}$$

where $\theta = \theta(x,t)$ and $\phi = \phi(x,t)$ and $\partial_x = \partial/\partial x$. The first term is the classical counterpart of exchange energy, the second and last terms are hard-axis and easy-axis anisotropies, respectively, which are of crystalline or demagnetizing origin. The constant Q^{-1} describes their relative strength. The motivation for the one-dimensional description (1) is the experimental evidence of planar symmetric static and moving domain walls in ferromagnetic films [3] and bulk ferromagnets [4] as well as the existence of effectively 1D magnetic materials [5]. In the presence of an external magnetic field, this model describes static confinement of two twisted π Bloch walls [6, 7] and nuclei for magnetization reversal [8].

In high purity ferromagnets damping is extremely small [4] and we are thus motivated to study the purely conservative time evolution $\partial_t\mathbf{M} = -\mathbf{M}\times\mathbf{H}_{\text{eff}}$ corresponding to the precession in the effective field $\mathbf{H}_{\text{eff}} = -\delta\mathcal{E}/\delta\mathbf{M}$ ($\delta/\delta\mathbf{M}$ is the functional derivative). In spherical coordinates, the equations of motion take the form

$$\sin\theta\,\partial_t\phi = \frac{\delta\mathcal{E}}{\delta\theta}, \qquad \partial_t\theta = -\frac{1}{\sin\theta}\frac{\delta\mathcal{E}}{\delta\phi}, \tag{2}$$

where $\mathcal{E}$ is given by (1). The model (2) has several interesting features. For large hard-axis anisotropy $Q^{-1}\to\infty$, the topology of the configuration space is tuned from a sphere into a circle while the dynamics (2) merges [7] into a sine-Gordon (SG) system in the easy-plane angle. Note that the latter system is relativistically invariant, in contrast to (2). Conversely, the present system may be considered as an extension of the SG system by allowing the field to "escape" into the third dimension along the hard axis. It is remarkable that this extension leads to a bound state of the formerly repelling SG soliton-soliton pairs. Note also that the energy density (1) is equivalent to that of the ubiquitious nonlinear O(3) σ model with anisotropies.

The system (2) has been proven to be integrable [9] but the evaluation of two-soliton solutions by the inverse scattering method proved to be extremely tedious even for simpler models [10]. Exact two-soliton solutions have been presented [11] but have not treated twisted soliton pairs with vanishing center of mass velocity. Hirota's formalism [12] has also been applied [13] and the results have been reviewed [14]. Unfortunately, the gyroscopic constraint relating the sign of the out of easy-plane component and the direction of motion has not been taken into account [15]. Since this constraint is crucial for the prediction of a bound state, we explicitly construct two-soliton solutions with vanishing center of mass velocity in the sequel.

The symmetries [16] of the equations of motion (2) allow us to parametrize solutions on the domain $0 \leq \phi < 2\pi$ and $0 \leq \theta < 2\pi$. Furthermore, we restrict ourselves to solutions with one specific set of boundary and initial conditions for θ and ϕ, respectively. Equivalent solutions are obtained by nontrivial symmetry transformations [16].

As a topological characterization of one- and two-soliton solutions with spatially uniform ϕ we introduce the twist

$$q(t) = \frac{1}{2\pi}\mathrm{sgn}[\phi(t) - \pi] \int_{-\infty}^{\infty} dx \frac{\partial \theta}{\partial x}(x,t), \tag{3}$$

where the parametrization of the solution is such that $0 \leq \phi < 2\pi$ and we define $\mathrm{sgn}(0) = +1$. Single solitons have $|q| = 1/2$. Soliton pairs with $q = 0$ have opposite relative twist whereas soliton pairs with $|q| = 1$ are twisted. Note, however, that q is in general not a topological invariant. The only topological invariant of soliton solutions of (2) is $\mathcal{Q} = \frac{1}{2}[M_z(x=\infty) - M_z(x=-\infty)]$. For a better understanding of two-soliton solutions, we briefly recall some properties of one-soliton solutions of (2).

One-soliton solutions.—They have the form [17] $\theta_K(\frac{x}{\delta} - \nu t)$, $\phi = \text{const}$ with

$$\theta_K(s) = 2\arctan(e^{-s}), \tag{4}$$

and hence $|q| = 1/2$. The constants δ, ϕ are connected via $\delta^{-2} = 1 + Q^{-1}\cos^2\phi$ and ν, ϕ obey the gyroscopic constraint (cf. Fig. 1)

$$\nu = (Q^{-1}/2)\sin 2\phi. \tag{5}$$

Note that here δ has arbitrary sign. For $\phi = \pm\pi/2, 0, \pi$, Eq. (5) describes static Bloch and Néel domain walls, respectively.

Since the equations of motion are not relativistically invariant, moving solutions cannot simply be obtained by boosting static solutions to a certain velocity. A physical mechanism different from Lorentz invariance is responsible for the motion of solitons. In a moving soliton, the magnetization at the center ($\theta = \pi/2$) of the soliton is tilted out of the easy plane. This hard-axis component

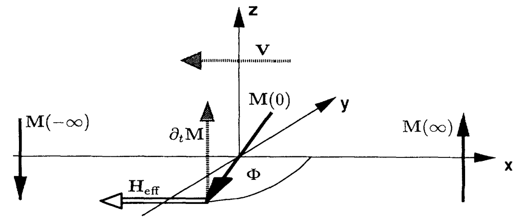

FIG. 1. Mechanism of motion of a single soliton. The out of easy-plane component of $\mathbf{M}$ creates an effective magnetic field $\mathbf{H}_{\text{eff}} = -Q^{-1}M_x\mathbf{e}_x$ at the soliton center. This field induces a torque on the magnetization and the soliton moves. Note that the sign of the velocity v is uniquely determined by the boundary conditions and the sign of $\sin 2\phi$.

induces a nonvanishing effective magnetic field in the x direction and causes the soliton to move (cf. Fig. 1).

The energy of a moving soliton (4) is given by

$$\mathcal{E}_K = 2/|\delta|, \tag{6}$$

where $0 < \delta^{-2} < 1 + Q^{-1}$. The lower and upper limits of (6) are attained for static Bloch and Néel walls, respectively. For low velocities of a Bloch domain wall ($\phi = \frac{\pi}{2} \pm \varphi, \varphi \ll 1$) it reduces to $\mathcal{E}_K = 2 + Q^{-1}\varphi^2$ showing that the hard-axis anisotropy can be considered as the kinetic energy of the soliton.

There are close analogies to a relativistic field theory: The maximum velocity $v_{\max} = \sqrt{1 + Q^{-1}} - 1$ of a soliton is finite and the width of a Bloch domain wall shrinks with increasing velocity. There is, however, an important difference: When the soliton reaches the maximum velocity $v_{\max}$, the energy (6) and hence the soliton kinetic energy remains finite. This is in fact the crucial point for the existence of a breather solution as described below.

Two-soliton solutions.—Consider a pair of domain walls (4) with width $\delta > 0$ centered at $x/\delta = \pm R$ with $\phi = \text{const}$, e.g., $\phi = \pi/2$. Arranging them such that they are untwisted, i.e., $q = 0$, we have

$$\theta_0 = \theta_K(R_0 - x/\delta_0) + \theta_K(x/\delta_0 + R_0), \tag{7}$$

whereas for a twisted pair of domain walls ($|q| = 1$) we write

$$\theta_1 = \theta_K(x/\delta_1 - R_1) + \theta_K(x/\delta_1 + R_1). \tag{8}$$

The configurations (7) and (8) are static structures only in the presence of an external magnetic field along the z axis [6–8]. Without an external field we expect the structures (7) and (8) to undergo a dynamic behavior. The energy of (7) and (8) with $\phi = \pm\pi/2$ is given by $\mathcal{E}_0 = 4\delta_0 \tanh^2 R_0$ and $\mathcal{E}_1 = 4\delta_1 \coth^2 R_1$, respectively. This is in accordance with the intuitive picture that untwisted domain wall pairs tend to attract each other, whereas twisted domain wall pairs repel.

Now we are going to show that the statically repelling domain walls can form a dynamically bound state under certain conditions. To this end we look at the simplest dynamic generalizations of (7) and (8). We put $\phi = \phi(t)$

and in (7) and (8) $R_i = R_i(t)$, $i = 0, 1$, or equivalently

$$\theta = 2\arctan\left[f(t)\Psi(x)\right], \qquad \phi = \phi(t), \tag{9}$$

with arbitrary f and Ψ. Inserting (9) into (2) we obtain

$$\partial_t f = Q^{-1} f \sin\phi\cos\phi, \tag{10}$$

$$-\partial_x^2\theta + K_{\text{eff}}(t)\sin\theta\cos\theta + h_{\text{eff}}(t)\sin\theta = 0, \tag{11}$$

where we have introduced a time dependent effective field $h_{\text{eff}}(t) = -\partial_t\phi > 0$ and a time dependent anisotropy $K_{\text{eff}}(t) = 1 + Q^{-1}\cos^2\phi$. The restriction in the sign of $\partial_t\phi$ is due to symmetry [16(ii)]. Equation (11) is identical in form to the Euler-Lagrange equations describing static structures in the presence of an external field. Integrating (11) with $\partial_x\theta$ at a fixed time we obtain the first integral

$$(\partial_x\theta)^2 + K_{\text{eff}}(t)\cos^2\theta + 2h_{\text{eff}}(t)\cos\theta = C(t). \tag{12}$$

Equation (12) has the form of an energy conservation and thus the qualitative behavior of soliton solutions can be extracted from a phase portrait discussion. For $C = K_{\text{eff}} - 2h_{\text{eff}}$ we obtain a solution that makes an excursion from $\theta = \pi$ returning to the same value. $C = K_{\text{eff}} + 2h_{\text{eff}}$ leads to a solution connecting two subsequent "potential maxima" at $\theta = 0$ and $\theta = 2\pi$. The former solution evidently belongs to $q = 0$, the latter to $|q| = 1$.

For $C = K_{\text{eff}} + 2h_{\text{eff}}$, Eq. (12) can easily be integrated. The consistency of the ansatz (9) requires the characteristic width $\delta_1 > 0$ to be time independent

$$\delta_1^{-2} = 1 - \partial_t\phi_1 + Q^{-1}\cos^2\phi_1 = \text{const.} \tag{13}$$

In order to meet our restriction $\partial_t\phi > 0$ we have $1 < \delta_1^{-2} < \infty$. The subscript indicates that this solution belongs to $|q| = 1$. Furthermore, we obtain $f = \delta_1^{-1}[-\partial_t\phi_1]^{-1/2}$. Together with (13), Eq. (10) is easily verified, showing the consistency of the ansatz (9). Integrating (12) and (13) and choosing $x = 0$ as the center of symmetry we obtain θ_1 as given in (8) but with time dependent R_1 and ϕ_1 which are parametrized by $\delta_1 > 0$:

I, $1 < \delta_1^{-2} < 1 + Q^{-1}$:

$$R_1(t) = \operatorname{arccosh}\left(\frac{1}{\nu\delta_1}\sqrt{1 - \delta_1^{-2} + Q^{-1}\cosh^2\nu t}\right),$$
$$\phi_1(t) = \arctan\left[\nu(\delta_1^{-2} - 1)^{-1}\coth\nu t\right], \tag{14}$$

where $\nu = [(\delta_1^{-2} - 1)(1 + Q^{-1} - \delta_1^{-2})]^{1/2}$.

II, $1 + Q^{-1} < \delta_1^{-2} < \infty$:

$$R_1(t) = \operatorname{arccosh}\left(\frac{1}{\omega\delta_1}\sqrt{\delta_1^{-2} - 1 - Q^{-1}\cos^2\omega t}\right),$$
$$\phi_1(t) = \arctan\left[\omega(\delta_1^{-2} - 1)^{-1}\cot\omega t\right], \tag{15}$$

with the frequency $\omega = [(\delta_1^{-2} - 1)(\delta_1^{-2} - 1 - Q^{-1})]^{1/2}$.

III, $\delta_1^{-2} = 1 + Q^{-1}$ separates the above cases:

$$R_1(t) = \operatorname{arccosh}\left(\sqrt{1+Q}\sqrt{1+Q^{-2}t^2}\right),$$
$$\phi_1(t) = \arctan\left(Qt^{-1}\right). \tag{16}$$

The branch of arctan is chosen such that $\phi_1(0) = \pi/2$ and such that $\phi_1(t)$ is continuous. The solutions (8) with (14)–(16) are defined on the extended domain $0 \le \theta < 2\pi$. Note that the solution II may be obtained from I by analytic continuation to values $\delta_1^{-2} > Q^{-1} + 1$.

Equations (8) and (14) describe the scattering of two twisted solitons. For $t \to -\infty$ they are converging from infinity, each with velocity $v_\infty = \nu\delta_1$, and establish an out of easy-plane component in accordance with (5). They are scattered at $t = 0$ with $R_1 = R_{\min} \equiv \operatorname{arctanh}\delta_1$ in a Bloch configuration, $\phi = \pi/2$. For $t \to \infty$ they return with reversed velocities to $x = \pm\infty$. This behavior is reminiscent of other nonlinear field theories.

Case II describes a bound state of a twisted soliton pair, a so-called breather state. The solitons oscillate between $R_{\min}$ and $R_{\max} \equiv \operatorname{arctanh}(\delta_1[1 + Q^{-1}]^{1/2})$ covering Bloch ($\phi = \pi/2, 3\pi/2$) and Néel ($\phi = 0, \pi$) configurations, respectively (see Fig. 2). The angle ϕ and the direction of soliton motion satisfy (5) for any t.

Case III is the limiting case corresponding to two initially resting solitons at infinity.

The total energy per unit area (1) of configurations I–III is given by

$$\mathcal{E}_1 = 4/\delta_1. \tag{17}$$

Comparing I and II, we recognize that the energy $\mathcal{E}_1$ of the $|q| = 1$ breather is always *higher* than that of the scattering solutions. This is a surprising fact and requires further explanation. The motion of soliton pairs is characterized by the conversion of exchange energy and soliton kinetic energy (alias hard-axis anisotropy energy) into each other. Let us consider a twisted soliton pair in a Bloch configuration, i.e., vanishing hard-axis anisotropy (cf. Fig. 2). Subjected to the equations of motion (2), the solitons initially separate from each other. For $\mathcal{E}_1 < 4\sqrt{1 + Q^{-1}}$, the angle ϕ is monotonically decreasing but it never reaches the hard-axis state ($\phi = 0, \pi$) and the solitons uniformly diverge to infinity. However, for $\mathcal{E}_1 > 4\sqrt{1 + Q^{-1}}$, a state of maximum hard-axis anisotropy is reached (cf. Fig. 2). Despite the

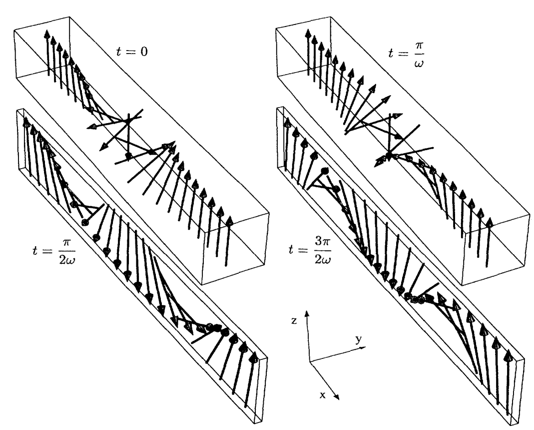

FIG. 2. Time evolution of the $|q| = 1$ breather (8) and (15).

fact that the exchange energy still would favor a further separation, the two solitons are confined in an energetically unfavorable state, periodically converting exchange and hard-axis anisotropy energy. Thus the boundedness of the kinetic energy is crucial for the existence of this type of breather. If the kinetic energy were unbounded, the excess in exchange energy could be completely converted into kinetic energy and the two solitons would not be bound.

It is instructive to compare this behavior with the dynamics of the $q = 0$ soliton pair. Since this result has already been quoted [11], we restrict ourselves to a short discussion. The dynamic solutions arise from the integration of (12) for $C = K_{\rm eff} - 2h_{\rm eff}$. They have the form (7) with $R_0 = R_0(t)$ and $\phi = \phi(t)$. Analogous to $|q| = 1$, three different cases have to be distinguished: For $1 < \delta_0^{-2} < 1 + Q^{-1}$ scattering solutions are obtained. For $0 < \delta_0^{-2} < 1$ breathing solutions exist and the limiting case $\delta_0 = 1$ corresponds to a scattering solution of two initially resting solitons. The energy is given by $\mathcal{E}_0 = 4/\delta_0$. In contrast to the $|q| = 1$ case, the lowest lying states are now the breathing states (cf. Fig. 3).

The existence of a bound state of a twisted soliton pair is not the only peculiarity of the present system. Naively one would expect that solitons converging from infinity with identical initial velocities would approach more closely if they form an untwisted pair than if they are twisted. Comparing the solution of (12) for $C = K_{\rm eff} - 2h_{\rm eff}$ with (14), we recognize that this is in general not true. A large initial hard-axis component of an untwisted soliton pair prevents a convergence of the constituents since the "kinetic" energy is already exhausted.

There are two possible experimental realizations of this surprising effect. Applying an external field along the z axis, the twisted domain wall pair (8) may be squeezed [7]. Switching off the external field, the domain walls may remain dynamically confined for a certain time provided that the initial excess in exchange energy was sufficiently large and damping sufficiently small. Second, Eq. (11) shows that the confinement is due to an effective field arising from the precession. Conversely, this precession may be maintained by a pulsed external magnetic field thus leading to a confinement of the domain walls.

However, for 3D ferromagnets, these scenarios are based on the assumption that the domain walls do not develop instabilities violating the premise of planar symmetry. Including shape demagnetizing effects and damping, this issue has been examined numerically [18] for small particles. The results support our assumptions: For strong pulses of the external field, the formation of Bloch lines is suppressed and domain walls propagate retaining their planar symmetry. Furthermore, the interaction of twisted and untwisted domain wall pairs may clearly be distinguished showing that the exchange interaction of domain walls dominates over demagnetizing induced repulsion [3].

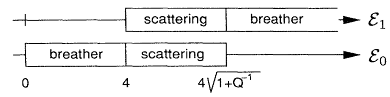

FIG. 3. Energy spectrum of two-soliton solutions with $V = 0$ for $|q| = 1$ and $q = 0$. Note that the $|q| = 1$ breather states always have larger energy than two independent solitons moving at arbitrary velocity.

Other expected candidates for such anomalous breather states are systems where the kinetic energy of solitons is bounded.

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[15] Equations (7.38–42) of Ref. [14] do not contain the gyroscopic constraint of our Eq. (5) (cf. also Fig. 1).

[16] Equation (2) has the following symmetries: (i) the trivial symmetry arising from the ambiguity of the mapping $\mathbf{M} \mapsto (\theta, \phi)$, (ii) rotations by π in spin space around the x, y, and z axis, and (iii) time reversal symmetry $(t, \mathbf{M}) \mapsto (-t, -\mathbf{M})$.

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