Kinetics of Systems with Continuous Symmetry under the Effect of an External Field

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Universality in the kinetics of systems with O(N) continuous symmetry, under influence of an external time-dependent field, is studied. The exponents that characterize the process are obtained, showing that they are independent of N, for $N \geq 2$. We also get analytical expressions for the time dependence of the order parameter (magnetization) in the presence of small fields, in the limit $N \to \infty$. Our results show that universality goes beyond the value of the exponents. For small fields, the full dynamical evolution of the order parameter is universal. Similar universality is expected for arbitrary N. Agreement with numerical solutions of the basic nonlinear equations is excellent.

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Universality is a concept that has always attracted the interest of physicists. The existence, in some physical processes, of parameters and functions that are independent of microscopic details has provided an area of sustained activity over more than thirty years. Typical examples are critical phenomena, where the correlation length grows to infinity, and kinetics of first-order phase transitions [1,2], where the system is characterized by the mean domain size that grows in time. Similarly, we can expect universality of some kind in any process where the characteristic length of the system becomes much larger than the range of interaction. An example of a process of this type is a magnetic system (below T_c) evolving under the effect of an external field *opposed* to the magnetization. The classical view of this dynamical process indicates that the system separates into domains, each one rotating its magnetization independently. When the field amplitude tends to zero, the domain size diverges; in this Letter, we analytically determine the universal features of this dynamical process. A proposal of universality for such a system was made by Rao, Krishnamurthy, and Pandit in [3], where the particular case of the hysteresis loop induced by a sinusoidal field, $H(t) = H_0 \sin \Omega t$, was studied. When H_0 and Ω are decreased (taking the limit $\Omega \to 0$ first, in order to guarantee that the system completes the loop) the total area W (energy dissipation) of the hysteresis loop also vanishes; so it was proposed [3] that

$$\lim_{H_0 \to 0} \lim_{\Omega \to 0} W \propto H_0^{\alpha} \Omega^{\beta}, \tag{1}$$

with α and β as universal exponents. Similar arguments can be applied in general to a system below T_c evolving under the effect of a variety of external fields opposed to the magnetization, including a step-function field. Another universal exponent η is also relevant: If one defines t_1 as the time when the magnetization (order parameter) changes sign, then

$$\lim_{H_0 \to 0} t_1 \propto H_0^{-\eta}. \tag{2}$$

In this Letter we first analyze the instability induced by the external field in a simplified one-mode model. We get, through a simple linear analysis, the exponents α, β, and η, showing that they are independent of the spatial dimensionality D and the order parameter dimensionality N, for $N \geq 2$. Once the role of the instability is described, we use Suzuki's ideas for systems near an instability point [4] to analytically obtain a closed expression for $dM(t)/dt$ in terms of $M(t)$ and $H(t)$ [for $H(t) \ll 1$]. An important consequence of the calculation is that the universality is not just for the exponents α, β, η, but the full equation for $dM(t)/dt$ is universal. This equation is independent of the particular form of the free energy and depends only on $H(t)$ and on the equilibrium value of the order parameter. The analytic results are compared with numerical calculations; an excellent agreement is found even for fields that are not very small. Our analysis is performed in the limit of large dimensionality of the order parameter, $N \to \infty$, but it provides strong support for similar universal features [possibly an N-dependent equation for $M(t)$] for arbitrary N. The universality of the exponents is solely due to the existence of a Goldstone mode and should be valid for all $N \geq 2$.

The framework used in theoretical and computational studies for various problems in kinetics of phase ordering and phase separation is the Langevin equation; we also begin our analysis with it for an N-component order parameter $\mathbf{\Phi}$ in a D-dimensional space $\mathbf{x}$, in which the thermal noise satisfies a fluctuation-dissipation relation for a system in equilibrium at temperature β^{-1}. We restrict ourselves to systems with a nonconserved order parameter. A large number of studies have used the Landau-Ginzburg-Wilson free-energy functional

$$F = \int d^D x \left[\frac{\kappa}{2} (\nabla \cdot \mathbf{\Phi})^2 + \frac{r}{2} \mathbf{\Phi}^2 + \frac{u}{4N} (\mathbf{\Phi}^2)^2 - \mathbf{H}(t) \cdot \mathbf{\Phi} \right], \tag{3}$$

where we have included the coupling term to a time-dependent external field $[\mathbf{H}(t)]_i = H(t) \delta_{i1}$. We assume that for $t \leq 0$, $H(t) \leq 0$ and $\langle \Phi_1(t) \rangle - M(t) \approx -M_0$, where M_0 is the absolute value of the equilibrium magnetization. A simplified picture of the process that we are interested in can be obtained by imagining that our system is a ball inside a Mexican hat (for $N=2$). For $t<0$ the system is at (or near to) equilibrium, but after $H(t)$ changes sign, at $t=0$, the absolute minimum of the free energy is placed at the opposite side of the hat. We see that during the very early time the ball rapidly (exponentially) finds equilibrium in the direction parallel to the field, reaching a saddle point. At this point the system is unstable in the perpendicular direction, with

$$-\gamma_0(t) \equiv \left.\frac{\partial^2 f(\Phi)}{\partial \Phi_\perp^2}\right|_{\text{saddle point}} = -\frac{H(t)}{M_0} \tag{4}$$

to lowest order in $H(t)$. Here $f(\Phi)$ is the bulk part of the free-energy density. This instability is common to all systems with $N \geq 2$ and is the general consequence of the ground state degeneracy that makes the Goldstone mode unstable in the presence of any nonzero field (opposed to the magnetization). It is also independent of the particular form of the free energy (so long as the ground state is degenerate over the surface of an N-dimensional hypersphere of radius M_0).

Linear analysis of this simple model shows that the instability induces, during the very early time, exponential growth of the form

$$M(t) \approx -M_0 + \delta \exp\left(\int_0^t dt' \gamma_0(t')\right), \tag{5}$$

where δ is a small parameter related to the distance to the (unstable) equilibrium point or to a mechanism that destroys the (unstable) equilibrium. In our case, assuming perfect alignment of the field with respect to the magnetization, δ is related both to the temperature and the field. A standard way to estimate t_1 in Eq. (2) is to relate it to the time when linear analysis becomes invalid ($\delta \exp[\int_0^t dt' \gamma_0(t')] \approx 1$). If at early times, $H(t) \propto h t^{n-1}$, then $t_1 \propto (-M_0 h^{-1} \ln\delta)^{1/n}$. This result shows that $\eta = 1/n$ and that, if δ depends on h, there are also logarithmic corrections. Also,

$$W = \int M(t) dH(t)$$
$$\approx 2M_0 h t_1^{n-1} \propto M_0^{(2n-1)/n} h^{1/n} (-\ln\delta)^{(n-1)/n}.$$

For example, for an oscillating square wave pulsed field of period T_0, we have $n=1$, obtaining $\alpha=1$, $\beta=0$, and $\eta=1$ (note that in the limit $\Omega = 2\pi/T_0 \rightarrow 0$ this parameter is irrelevant). For a sinusoidal field (for $\Omega \rightarrow 0$, in the region of interest $H_0 \sin\Omega t \approx H_0 \Omega t$), $n=2$ and $\alpha=\beta=\eta=\frac{1}{2}$.

The physical picture just presented is a "mean field" one in the sense that spatial fluctuations in the free energy are suppressed. In practice, for our case, spatial and temperature-induced fluctuations are essential; thus it is important to verify the previous results in a more rigorous way. Suzuki has proposed a singular perturbation method [4] to study systems near an instability point. We apply his ideas and their generalization to systems with infinite modes [5,6] to our problem.

An analytical treatment of the Langevin equation for arbitrary N is extremely difficult, but the previous picture suggests that the N dependence (if any) should be small. An important simplification can be performed in the limit $N \rightarrow \infty$ where a closed set of equations is available [7]. After properly scaling the units of time, length, and energy,

$$\frac{dM(t)}{dt} = \frac{1}{2}[\zeta_\perp(t) M(t) + H(t)], \tag{6}$$

$$\zeta_\perp(t) = [M_0^2 - M^2(t) + S_0 - S(t)] K[M,S], \tag{7}$$

$$S(t) = K_D \int_0^\Lambda dq\, q^{D-1} C_\perp(q,t), \tag{8}$$

$$\frac{\partial C_\perp(q,t)}{\partial t} = \epsilon - [q^2 - \zeta_\perp(t)] C_\perp(q,t), \tag{9}$$

where $C_\perp(q,t)$ is the Fourier transform of the transverse correlation function. Equation (7) corresponds to the general expression for an arbitrary free energy. $K[M(t), S(t)]$ can be a complicated function that vanishes at the metastable or unstable extrema of the free energy, but at the (degenerate) absolute minima, it is the expression in the square brackets that vanishes [for the free energy (3), $K(M,S)=1$]. In these equations, a large-q cutoff Λ is introduced, ϵ is the scaled temperature ($\epsilon = u/\beta\kappa^{D/2}$), S_0 is the equilibrium value of $S(t)$, and K_D is proportional to the surface area of the unit hypersphere in D-dimensional space. These equations are the starting point of our analysis. The main ingredient lost in the $N=\infty$ model compared with real systems ($N=2$, $N=3$) is the existence of topological defects. Those are important in ordering dynamics [6]; in our problem these will be important for late times but should not modify the main conclusions of our analysis.

The similarity between Eqs. (6)-(9) and the mean-field-like picture is clear. When $H(t)$ changes sign, after a short time, $dM(t)/dt \approx 0$, $\zeta_\perp(t) \approx -H(t)/M_0$, and Eq. (9) becomes unstable [note that the Goldstone mode $q=0$ in Eq. (9) has precisely the exponential growth predicted by the "mean field" picture] and Suzuki's approach should be applicable. A technical problem, which makes a singular perturbation analysis difficult, is that both $\gamma_0(t)$ and δ in Eq. (5) are field dependent. It is convenient to separate these two effects by introducing a change of variables which makes the argument of the exponential in Eq. (5) independent of the amplitude of the external field (to lowest order). Let us assume that near $t=0$, $\int dt\, H(t) \approx h t^n$, then it is clear that the following transformation is useful:

$$\tau = h^{1/n} t,\quad F(\tau) = h^{-1/n} H(t),\quad Q^2 = h^{-1/n} q^2, \tag{10}$$

$$a(\tau) = h^{-1/n} \zeta_\perp(t),\quad c(Q,\tau) = h^{D/2n} C_\perp(q,t). \tag{11}$$

After such a transformation, the parameter δ is easily recognized as $\delta = \epsilon h (D-2)/2n$ (the initial condition could also depend on this parameter). If the particular form of $F(\tau)$ is known, it is possible to perform a singular perturbation analysis (in δ) in order to get an asymptotic solution for $\tau \gg 1$ when $\delta \rightarrow 0$ [8]. Instead, we perform a more general analysis for an arbitrary field $F(\tau)$, treating together two types of physical situations: (a) At time $t=0^-$ the system is in equilibrium [with $M(0) = -M_0, c(Q,0) = \epsilon h^{(D-2)/2n}/Q^2$] and then a field is turned on. (b) There is present (since time $t=-\infty$) a periodic field $H(t+T_0)=H(t)$ and the system undergoes a hysteresis cycle. In each of the two cases, it is of course the time dependence of $H(t)$ near $t=0$ that determines the exponent n in the transformation Eqs. (10) and (11).

The strategy of the general analysis has three steps. First, we formally integrate the equation for $c(Q,\tau)$. And, we integrate $c(Q,\tau)$ to obtain $S(\tau)$ [9]:

$$S(\tau)=S_0+K_D\epsilon h^{(D-2)/2n}\int_0^{\Lambda/h^{1/2n}} dQ\,Q^{D-3}\exp\left[-Q^2\tau+\int_0^\tau d\tau''a(\tau'')\right]\int_{\tau_0}^{\tau} d\tau' a(\tau')\exp\left[Q^2\tau'-\int_0^{\tau'} d\tau''a(\tau'')\right],$$

where τ_0 is either 0 or $-\infty$ depending on whether the case being considered is (a) or (b). Second, we formally get the asymptotic behavior for $\tau\gg 1$ and $\int_0^\delta d\tau' a(\tau')\gg 1$ but with $\delta\to 0$ [9]. The key idea is to realize that the points in the plane (τ',Q) that most contribute to the double integral are those with $\tau'\approx 0$ (the fastest exponential growth) and $Q\approx 0$ (the most unstable mode). It is enough, for our purpose, to construct a differential equation for $S(\tau)$ for $\tau\gg 1$; we obtain [9,10]

$$\frac{dS(\tau)}{d\tau}\approx[a(\tau)-f(\tau)][S(\tau)-S_0]\,, \qquad (12)$$

with $f(\tau)=(D-2+2j_0)/2\tau+O(\tau^{-2})$, where j_0 can be 0 or 1 depending on the type of external field [10]. Note that in deriving this equation we did not make use of the explicit form of the free energy. This information is contained in the parameter $a(\tau)$ [see Eqs. (7), (10), and (11)].

Finally, in the third step, we construct a closed equation for $M(\tau)$ solving the system of Eqs. (6), (7), and (12) in the limit $h\to 0$. In this limit Eq. (7), for an arbitrary free energy, reduces to $M_0^2-M^2(\tau)=S(\tau)-S_0+O(h^{1/n})$. This fact shows that the final result will be universal, independent of the form of the free energy (as long as the system is below T_c). The physical reason underlying universality is that, as $h\to 0$, the characteristic length of the process diverges and fluctuations affecting $M(t)$ inside each domain partially cancel. As a result, in the renormalization group sense, the system approaches a fixed point, the zero-temperature fixed point. The universal solution is [9]

$$M(\tau)=M_0\tanh\left[\frac{\int_{\tau_1}^{\tau}d\tau' F(\tau')}{2M_0}-g(\tau)\right], \qquad (13)$$

where $g(\tau_1)=0$ and $g(\tau)$ satisfies the differential equation

$$\frac{dg(\tau)}{d\tau}=-\frac{1}{2}f(\tau)\tanh\left[\frac{\int_{\tau_1}^{\tau}d\tau' F(\tau')}{2M_0}-g(\tau)\right]. \qquad (14)$$

This equation is easily solved iteratively. The only unknown quantity is τ_1 which is defined from $M(\tau_1)=0$. Following Suzuki's approach, τ_1 can be uniquely fixed by matching this asymptotic solution with the linear analysis in the parameter δ. Both the linear approximation (for $\tau\gg 1$) and Eq. (13) (for $\tau\ll\tau_1$) yield the same form:

$$M(\tau)=-M_0+C\tau^{-[(D-2+2j_0)/2]}\exp\left[\int_0^\tau d\tau'\frac{F(\tau')}{M_0}\right]. \qquad (15)$$

Relating the constants C of the two approaches we get τ_1 in terms of h. In particular, for a step-function field ($n=1$) starting from equilibrium,

$$\epsilon h^{(D-2)/2}=\frac{8M_0^2}{K_D\Gamma[(D-2)/2]}\exp\left[-\frac{\tau_1}{M_0}-2g(1)\right], \qquad (16)$$

or, substituting $D=3$, $\tau_1=ht_1$, and neglecting the second term in the exponential,

$$t_1\simeq h^{-1}M_0\ln\left(\frac{16\pi\sqrt{\pi}M_0^2}{\epsilon\sqrt{h}}\right),$$

which verifies the results of the one-mode model.

For a sawtooth (or sinusoidal) periodic field ($h=H_0\Omega$, $n=2$) we get

$$\epsilon h^{(D-2)/4}=\frac{8M_0^{3/2}}{K_D\Gamma(D/2)\sqrt{2\pi}}\exp\left[-\frac{\tau_1^2}{2M_0}-2g(1)\right], \qquad (17)$$

leading to $t_1\simeq h^{-1/2}[2M_0\ln(16\pi\sqrt{2}M_0^{3/2}/\epsilon h^{1/4})]^{1/2}$.

Equations (13) and (14) and (16) and (17) are the central result of this Letter [note that one can also get universal expressions for $C(Q,\tau)$ and $a(\tau)$]. We have also performed a direct numerical integration of Eqs. (6)–(9) with $D=3$, for a step-function field and a periodic sawtooth field, in order to test Eqs. (16) and (17). The q integral was cut in two asymmetric pieces [partially taking into account the asymmetric behavior of $C(q,t)$] and each piece integrated with 16- or 32-point Gaussian quadratures. The resulting set of coupled differential equations was solved using a fourth-order Runge-Kutta algorithm. The results for the step-function field are shown in Fig. 1. Here $h=H_0$, the step-function jump in the field. The main figure shows a comparison (in a log-log plot) for t_1/M_0 vs H_0, the exponent η is directly obtained from the slope of the lines. The inset shows t_1H_0/M_0 vs H_0. The logarithmic correction is obtained from the slopes in the inset; we see that both numerical and analytical data are approaching the same slope for $H_0\to 0$.

For the periodic sawtooth field the numerical integration was iterated until a perfect periodic loop was obtained. Also the period of the field $H(t)$ was increased until no dependence on it remained (typically $T_0\approx 20h^{-1/2}$). In Fig. 2, the results for the area W of the hysteresis loop are shown. The theoretical results were obtained from the magnetization profile given by Eqs.

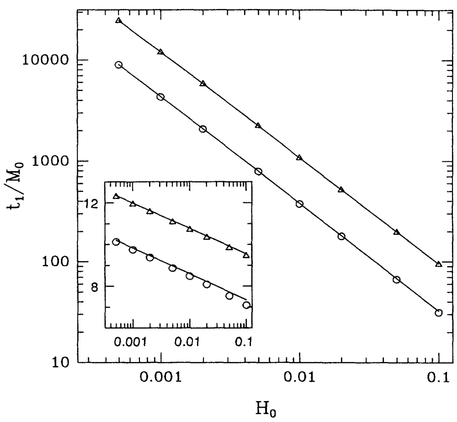

FIG. 1. Log-log plot of t_1/M_0 vs H_0 [defined in Eq. (2)] for $r=-2$ and $r=-10$. The symbols correspond to the numerical calculation (△, $r=-10$; ○, $r=-2$) and the solid lines to the analytical model [Eqs. (13), (14), and (16)]. The inset shows t_1H_0/M_0 vs H_0 (on a logarithmic scale), which displays the logarithmic corrections to the power law behavior of t_1.

(13), (14), and (17). Again agreement is excellent and increasingly better as $h \to 0$.

We are aware of a recent qualitative calculation [11] which gives similar exponents (also with logarithmic corrections) for the hysteresis loop of a $O(N=\infty)$ system. We have performed a more rigorous and fully analytic derivation, which allows us to recognize the underlying reasons for the particular exponents obtained and argue that the same values should remain for any system with $N \geq 2$ (in agreement with preliminary simulations for a three-dimensional system with $N=2$ and $N=3$ [12]). Moreover the fully analytic calculation permits us to recognize that universality is stronger than expected and is not restricted to the exponents.

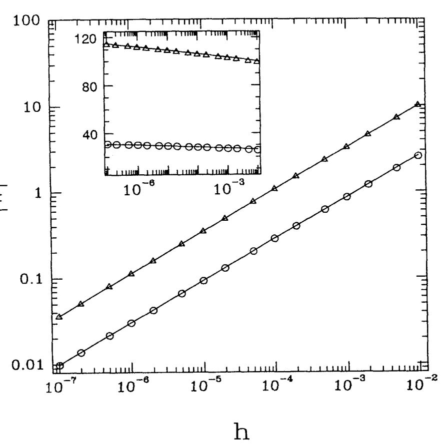

FIG. 2. Log-log plot of the area of the hysteresis loop for a periodic sawtooth field vs h (see text). The symbols are the same as in Fig. 1. The inset shows $|W|/h^{1/2}$ vs h (on a logarithmic scale), and displays the logarithmic corrections to the power law behavior of W [see Eq. (1)].

In summary, we have applied for the first time Suzuki's ideas to a system with infinite modes and spatial as well as thermal fluctuations. The analytic results were used to investigate universality in a system with continuous symmetry under the effect of an external field (sinusoidal and step-function periodic fields were considered). We used simple arguments to demonstrate the universality of the exponents defined in Eqs. (1) and (2) and to derive their universal values. We also analytically calculated $M(t)$ for the limit $N \to \infty$. We showed that the full dynamical process is dominated by the zero-temperature fixed point and thus $M(t)$ is universal. For the same physical reasons we expect the identical universality for any $N \geq 2$. Experimental measurements in the limit $h \to 0$ would be interesting.

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