Onset of the Sawtooth Crash

F. L. Waelbroeck [(a)]
Laboratory for Plasma Research, University of Maryland, College Park, Maryland 20742-3511
(Received 22 February 1993)

A new class of $m=1$ magnetic islands is presented. These islands have bounded current-density distributions, so that their resistive evolution is slow. Conservation of the total magnetic flux, however, results in a critical width such that the current density diverges and the separatrix collapses at the X point.

PACS numbers: 52.30.Bt, 52.30.Jb, 52.35.Py, 52.55.Fa

The role that magnetic reconnection plays in the sawtooth crash has become unclear [1,2]. A resistive instability with poloidal mode number $m=1$ is known to occur under very general conditions in tokamaks with safety factors less than unity [3]. The nonlinear behavior of this instability, however, is incompletely understood.

By nature, sawtooth oscillations require a well-defined transition between a slow and a rapid mode of evolution. The $m=1$ instability is known to have a rapid mode of growth, the kink-tearing mode [4,5]. In this Letter, I will show that it also has a distinct slow mode of growth, the $m=1$ tearing mode. I will further show that a transition to the kink-tearing mode occurs at a critical island width.

By virtue of Ohm's law, slow or diffusive growth requires that the perturbed current density $\tilde{J}$ be commensurate with the equilibrium current density J_0. The $m=1$ tearing mode is similar in this respect to $m \geq 2$ tearing modes [6]; both have homogeneously distributed current, unlike the kink-tearing mode [7]. The amplitude of the current-density perturbation, however, is significantly larger for the $m=1$ than for the $m \geq 2$ tearing modes. For thin islands ($w \ll \lambda$, where λ is the wavelength), $\tilde{J}$ may be evaluated from the property that the current perturbation is proportional to the amount of reconnected flux $\tilde{\psi}$:

$$\int \tilde{J}\, dx = \Delta' \tilde{\psi} ,$$

where x is the transverse coordinate. The constant of proportionality Δ' measures the free energy available for reconnection. Upon substitution of the estimate $\tilde{\psi} \sim J_0 \times w^2$, one finds

$$\tilde{J}/J_0 \sim w\Delta' .$$

It is well known that the $m=1$ instability is distinguished by large positive values of Δ' [8],

$$\lambda \Delta'_1 \gg 1 .$$

As a result, the classical analysis of tearing modes [6], based on the approximation that the flux is constant throughout the island region, holds only for extremely thin $m=1$ islands ($w \ll 1/\Delta'_1$). This can be seen from $\tilde{\psi}(w) - \tilde{\psi}(0) \sim \tilde{J} w^2 \sim (w\Delta')\tilde{\psi}(0)$. The regime $w\Delta'_1 \sim 1$, by contrast, is marked by strong current nonlinearities and requires the application of a different formalism.

Before describing the regime $w\Delta'_1 \sim 1$, it is necessary to comment on the role played by the braiding of magnetic field lines. Magnetic braiding has been invoked to explain, most notably, observations of sawtooth crashes without complete reconnection of the core magnetic flux [9]. However, significant braiding requires fairly large islands. The work presented here, by contrast, is concerned with thin islands. This work and the magnetic braiding model should thus be regarded as complementary.

The analysis proceeds from the low-beta reduced magnetohydrodynamic (MHD) equations. The magnetic field is taken to be helically symmetric. It is expressed in terms of the helical flux ψ through the auxiliary field $\mathbf{B}_*$:

$$\mathbf{B} = \hat{\mathbf{z}} + \epsilon (r\hat{\boldsymbol{\theta}} - \mathbf{B}) ,$$

where $\mathbf{B}_* = \hat{\mathbf{z}} \times \nabla\psi$. Here ϵ is the inverse aspect ratio and $\hat{\mathbf{z}}, \hat{\boldsymbol{\theta}}$ are, respectively, the toroidal and poloidal unit vectors.

The current density J is determined by Ampère's law, $J = \nabla^2\psi - 2$. In equilibrium, $\mathbf{B}\cdot\nabla J = 0$ so that J is constant on flux surfaces:

$$\nabla^2 \psi = I(\psi) .$$

For thin islands, the Laplacian can be approximated by $\partial^2\psi/\partial x^2$, where $x = r - r_s$ and r_s is the radius of the $q=1$ resonant surface. A closed-form solution can then be obtained by successive radial integrations [7,10].

The first integration yields an expression corresponding to the Biot-Savart law for the longitudinal component of the auxiliary field,

$$B_{*\theta} = \pm \{2[F(\psi) - G(\theta)]\}^{1/2} , \quad (1)$$

where

$$F(\psi) = F(\psi_0) + \int_{\psi_0}^{\psi} I(\hat{\psi}) d\hat{\psi} .$$

Here ψ_0 is the value of the helical flux on the island's magnetic axis. The two solution branches in Eq. (1) are connected at the turning point where $B_{*\theta} = 0$, or $F(\psi_t) = G(\theta)$ (Fig. 1). $G(\theta)$ is thus the current contained within the flux surface ψ_t tangent to the radial chord intersecting the island at θ. The separatrix is the last flux surface to have a turning point: $F(\psi_s) = \max[G(\theta)]$. The constant of integration in F and G will henceforth be fixed by $G(0) = \max[G(\theta)] = 0$.

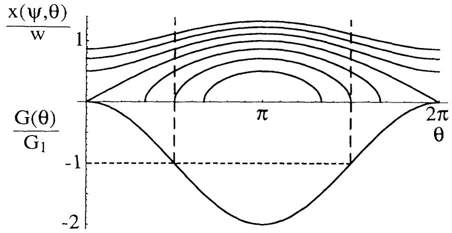

FIG. 1. Current well G and corresponding flux surfaces, with two radial chords.

The shape of the flux surfaces is determined by a second integration:

$$x_{\pm}(\psi,\theta)=x_t(\theta)\pm\int_{\psi_t}^{\psi}\frac{d\psi}{B_{*\theta}}\,, \tag{2}$$

where $x_t(\theta)$ represents the radial position of the turning point. The radial excursion $x_{\pm}(\psi,\theta)$ may alternatively be expressed as

$$x_{\pm}(\psi,\theta)=x_t(\theta)\pm\int_{\psi_t}^{\psi}\frac{dB_{*\theta}}{I(\psi)}\,. \tag{3}$$

Equation (3) shows that the radial excursion of a flux surface is the chord integral of the inverse current density (Fig. 1). The island equilibrium problem is thereby related to the problem of inverting chord-integrated tokamak measurements. Specifically, one finds from Eqs. (1) and (3) that the width of the separatrix is given by the Abel transform of the inverse current density,

$$w(G)=2^{1/2}\int_G^0 dF(F-G)^{-1/2}I(F)^{-1}\,, \tag{4}$$

where F and G serve as the radial and azimuthal independent variables. Abel's inversion formula will be used to construct a solution of the equilibrium equation.

The equilibrium problem is closed by matching the solution in the island region, Eq. (2), to the solution of the linearized toroidal MHD equation away from the island. There follows the global equilibrium equation

$$\int_{\psi_t}^{\infty}\frac{d\psi}{B_{*\theta}}-\int_{\psi_0}^{\infty}d\psi\left\langle\frac{1}{B_{*\theta}}\right\rangle=-\frac{\Delta_1' G_1}{2I_0^2}\cos\theta\,, \tag{5}$$

where

$$\langle h(\psi,\theta)\rangle=\frac{1}{2\pi}\int_{-\theta_t}^{\theta_t}d\theta\, h(\psi,\theta)$$

and θ_t is the turning point determined implicitly by $G(\theta_t)=F(\psi)$. Equation (5), as well as similar equations below, must be interpreted as applying to the limit where the upper bound of integration, taken to be *identical* for both integrals, goes to infinity. In this equation the first integral is the distance between two surfaces of equal flux, the second integral subtracts the $m=0$ component from the first, and the right-hand side represents the displacement of the core. The lack of an $m=0$ component in the equilibrium equation reflects the freedom to choose the initial current distribution. I will return to this point below.

A solution of the global equilibrium equation can be constructed as follows. First, a simplified expression for the $m=0$ component is obtained by evaluating Eq. (5) for $G=0$:

$$\int_{G_{\min}}^{\infty}\frac{dF}{I}\left\langle\frac{1}{B_{*\theta}}\right\rangle-\int_0^{\infty}\frac{dF}{(2F)^{1/2}I}=\frac{\Delta_1' G_1}{2I_0^2}\,. \tag{6}$$

The width of the separatrix is next calculated from the equilibrium equation:

$$w(G)=-2\int_0^{\infty}\frac{dF}{I}\left[\frac{1}{B_{*\theta}}-\frac{1}{(2F)^{1/2}}\right]-\frac{\Delta_1' G_1}{I_0^2}[\cos\theta(G)-1]\,. \tag{7}$$

Last, the inner flux distribution is found by inversion of Abel's equation, Eq. (5):

$$\psi(F)=\psi_s-\frac{1}{\pi}\int_F^0 dG\frac{w(G)}{[2(G-F)]^{1/2}}\,. \tag{8}$$

Equations (7) and (8) show that the current distribution *inside* the separatrix, $I(F)=(d\psi/dF)^{-1}$, is uniquely determined by the current distribution *outside* the separatrix and by the azimuthal dependence of the perturbation.

It is clear from Eqs. (7) and (8) that there exist $m=1$ islands with bounded current distributions. I will next show, however, that a current singularity invariably appears as a result of the Ohmic evolution of these islands. Whereas the complete solution of the evolution problem is outside the scope of the present Letter, the formation of the current sheet can be deduced from a global constraint resulting from Ohm's law. This constraint takes its simplest form in the case where the bulk current distribution is in diffusive equilibrium, $\eta J_0=E_0$. In this case, there is no diffusion of flux away from the island, and the total flux through the island region must be conserved. Equivalently, the $m=0$ component of the island layer width is conserved. Taking the circular equilbirium as a reference state, there follows

$$\int_{\psi_0}^{\infty}d\psi\left\langle\frac{1}{B_{*\theta}}\right\rangle=\int_0^{\infty}\frac{d\psi}{B_{*\theta0}}\,. \tag{9}$$

Note that this global flux-conservation property should not be confused with the local, detailed flux-conservation property characterizing the kink-tearing regime [5,7].

A more perspicuous form for the flux-conservation constraint may be obtained by writing, without loss of generality,

$$I(F)=I_0[1+A\iota(F/F_w)]^{-1} \tag{10}$$

for $F>0$, where $\iota(0)=1$ and A,F_w are the amplitude and width of the exterior current perturbation. Note that

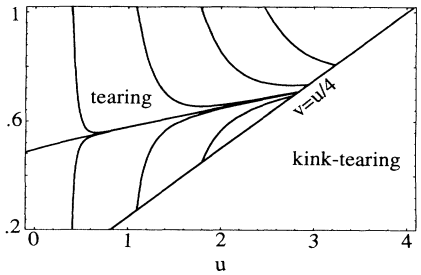

FIG. 2. Evolution of the relative current width v as a function of the island width u. The line $v=u/4$ represents the kink-tearing threshold.

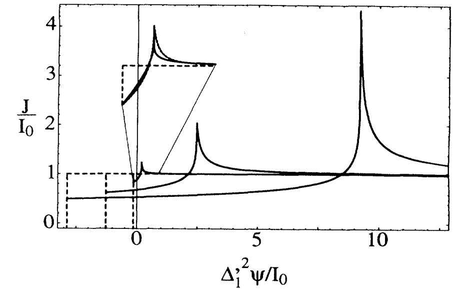

FIG. 3. Current (J/I_0) as a function of flux ($\Delta_1'^2\psi/I_0$) for $u=0.8$, 2.4, and 4.0. Inset: Comparison of the current distribution for the smallest island (dashed line) to the exact solution of Ref. [7] (solid line).

$A=-1$ represents current distributions with a singularity on the separatrix. Substituting Eq. (10) into Eq. (9) and using Eq. (6) yields

$$A=-\Delta_1' G_1/I_0(2F_w)^{1/2}\bar{\iota}, \tag{11}$$

where $\bar{\iota}=\int_0^\infty df f^{-1/2}\iota(f)$.

Assuming that $\bar{\iota}$ remains bounded during the island evolution, Eq. (11) shows that the width of the current distribution, F_w, must grow like the square of the mode amplitude G_1 if a current singularity is to be avoided. From simple scaling considerations, however, one expects the current width to grow in direct proportion to the mode amplitude. The formation of current singularities is thus unavoidable.

The slope of the separatrix at the X point is $2(1+A)\times[-G''(0)]^{1/2}/I_0$, where $G''=d^2G/d\theta^2$. The separatrix crossing angle therefore vanishes for $A=-1$. This supports the interpretation of the critical amplitude as a threshold for the kink-tearing mode, the X point being replaced by a current ribbon in this mode.

It is useful to illustrate the above considerations with a particular model. I will take $G(\theta)=G_1(\cos\theta-1)$, with $G_1>0$, and $\iota(f)=(1+f)^{-3/2}$, $f>0$. The criticality condition is then

$$G_1\le 2I_0(2F_w)^{1/2}/\Delta_1'. \tag{12}$$

The solution of the equilibrium equations, Eqs. (7) and (8), can be implemented analytically for the above model. The evolution of the island may then be studied phenomenologically by applying Ohm's law to the separatrix and to the island's magnetic axis. I have integrated numerically the resulting dynamical equations for the variables $u=\Delta_1' G_1^{1/2}/I_0$ and $v=(F_w/2G_1)^{1/2}$, measuring, respectively, the width of the island and the *relative* width of its current distribution. The trajectories are shown in Fig. 2. The integration was stopped at the critical amplitude $v=u/4$. The most significant feature is the attracting trajectory corresponding to the initial condition $v(0)=v_0\cong 0.497$ and $u(0)=0$. For sufficiently thin islands, $u\ll 1$, the attracting solution is self-similar: $\dot{v}=0$. This is not evident in Fig. 2 as a result of the singularity of the evolution for $u\ll 1$: $\dot{v}\sim(v-v_0)/u^2$. This singularity translates the property that arbitrarily peaked currents diffuse arbitrarily fast; only when the current has relaxed, after a time $t>w^2/\eta$, does the evolution proceed at the slower rate $\dot{w}\sim\eta\Delta'$.

The phenomenological evolution model is validated in part by the agreement between the predicted value of the reconnection rate in the self-similar limit, $\dot{u}\cong 0.25$, and the exact value given by Rutherford [6]. The current distributions for three islands lying on the attracting trajectory are shown in Fig. 3. The smallest island is near the self-similar limit; its current distribution is compared to the exact solution in the inset. The agreement is again excellent. Figure 4 shows the separatrix for the same three islands and for the critical island.

I have shown that well-defined sawtooth crashes result from the toroidal effects embodied in Δ_1'. This is consistent with numerical simulations of complete sawteeth: In toroidal geometry [11], a clear, heretofore unexplained jump in the growth rate follows long periods of sluggish growth. In reduced magnetohydrodynamics, by contrast, where $(\Delta_1')^{-1}=0$, sawteeth are rounded as the onset depends on discontinuities in the gradients of equilibrium quantities such as the temperature and safety factor [12]. In cylindrical geometry, last, $\lambda\Delta_1'\ll -1$ and the $m=1$ mode is ideally unstable. Nonlinear saturation results in

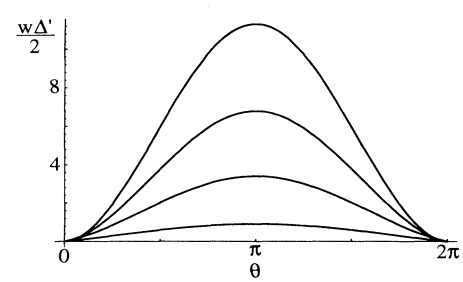

FIG. 4. Half width of the separatrix for the islands of Fig. 3 and for the critical island.

current sheets and thus leads to the kink-tearing regime; the tearing regime is never encountered [13].

Comparison with experimental observations requires a knowledge of the evolution of Δ_1' during the sawtooth ramp. On the basis of the neoclassical results in Fig. 3 of Ref. [8], I estimate the value of Δ_1' after a typical JET sawtooth ramp to be 2.5 cm^{-1}. Using the value $w_{\mathrm{crit}}\Delta_1' \cong 25$ obtained from Fig. 4 of this paper, there follows $w_{\mathrm{crit}} \cong 10$ cm. This is comparable to the widths of precursor [14] and snake [15] oscillations. It is too large, however, to account for sawteeth *without* measurable precursors, the present detection limit being about 1 cm.

It is instructive to compare the criterion $w_{\mathrm{crit}}\Delta_1' \cong 25$ to linear results. In linear theory, the transition between the tearing mode (with growth rate $\gamma \sim k_\parallel'^{2/5}\eta^{3/5}\Delta'^{4/5}$) and the kink-tearing mode ($\gamma \sim k_\parallel'^{2/3}\eta^{1/3}$) occurs for $w_i\Delta' \cong 1$, where $w_i = \eta^{1/3}k_\parallel'^{1/3}$ is the inertial tearing layer width [16]. Here $k_\parallel'$ is the radial derivative of the parallel component of the wave vector and η is the resistivity. The parameter $w_i\Delta_1'$, however, depends exclusively on slowly evolving global equilibrium parameters, unlike $w\Delta_1'$, and thus cannot account for the abruptness of the onset. Nevertheless, the linear threshold appears to be consistent with experimental observations [17].

It is a pleasure to acknowledge helpful discussions with J. F. Drake, A. W. Edwards, J. M. Finn, and C. S. Liu. This work was supported by a fellowship from the Center for Theoretical Physics at the University of Maryland and by the U.S. Department of Energy.

[a] Present address: Institute for Fusion Studies, University of Texas, Austin, TX 78712.

[1] A. W. Edwards *et al.*, Phys. Rev. Lett. **57**, 210 (1986).
[2] Y. Nagayama *et al.*, Phys. Rev. Lett. **67**, 3527 (1991).
[3] R. J. Hastie *et al.*, Phys. Fluids **30**, 1756 (1987).
[4] B. B. Kadomtsev, Fiz. Plazmy **1**, 710 (1975) [Sov. J. Plasma Phys. **1**, 389 (1975)].
[5] A. Y. Aydemir, Phys. Fluids B **4**, 3469 (1992).
[6] P. H. Rutherford, Phys. Fluids **16**, 1903 (1973).
[7] F. L. Waelbroeck, Phys. Fluids B **1**, (1989).
[8] C. G. Gimblett *et al.*, in *Proceedings of the Eighteenth European Conference on Controlled Fusion and Plasma Physics, Berlin, 1991* (European Physical Society, Petit-Lancy, Switzerland, 1991), Vol. II, p. 21.
[9] A. J. Lichtenberg, K. Itoh, S.-I. Itoh, and A. Fukuyama, Nucl. Fusion **32**, 495 (1992).
[10] I. B. Bernstein, J. M. Greene, and M. D. Kruskal, Phys. Rev. **108**, 71 (1957).
[11] A. Y. Aydemir, D. W. Ross, and J. C. Wiley, Phys. Fluids B **1**, 774 (1989); T. Sato *et al.*, Phys. Rev. Lett. **63**, 428 (1989).
[12] R. E. Denton, J. F. Drake, R. G. Kleva, and D. A. Boyd, Phys. Rev. Lett. **56**, 2477 (1986).
[13] H. Baty, J.-F. Luciani, and M.-N. Bussac, Nucl. Fusion **31**, 2055 (1991).
[14] P. A. Duperrex, A. Pochelon, A. W. Edwards, and J. A. Snipes, Nucl. Fusion **32**, 1161 (1992).
[15] A. Weller *et al.*, Phys. Rev. Lett. **59**, 2303 (1987).
[16] G. Ara *et al.*, Ann. Phys. (N.Y.) **112**, 443 (1978).
[17] S. Migliuolo, F. Pegoraro, and F. Porcelli, Phys. Fluids B **3**, 1388 (1991).