Reynolds Number Dependence of Isotropic Navier-Stokes Turbulence

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Reynolds number dependence of turbulence energy spectra and higher-order moments of velocity differences is explored by numerical integrations of the incompressive Navier-Stokes equation. The simulations have spatial resolutions up to 512^3 and cover $15 \leq R_\lambda \leq 200$, where R_λ is the Taylor microscale Reynolds number. The energy spectra collapse when scaled by the wave number k_p of peak dissipation and also by the spectrum level at k_p. k_p varies with R_λ in accord with the 1941 Kolmogorov theory. High-order normalized moments of velocity differences over inertial-range distances exhibit an R_λ-independent variation with separation distance. Implications of these observations are discussed.

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Fully developed turbulence tempts the theorist to search for statistical properties that are universal when examined in terms of the correct variables. Kolmogorov [1] offered such a theory of universal statistics over a half century ago. He postulated that, in the limit of vanishing viscosity, the mean dissipation rate ϵ becomes independent of viscosity ν. In addition, he assumed that ϵ is the only physical parameter defining the statistics of turbulent fluctuations at scales small compared with those that contain most of the kinetic energy but large compared with those on which viscosity acts strongly. This yields the famous prediction of an inertial-range energy spectrum $E(k) \sim k^{-5/3}$, where k is the wave number. The 1941 Kolmogorov theory also predicts a characteristic dissipation wave number, $k_d = (\epsilon/\nu^3)^{1/4}$, in terms of which turbulence energy spectra of the small scales take the isotropic form

$$E(k) = E(k_d)F(k/k_d)\,, \qquad (1)$$

where $F(x)$ is a universal function.

The 1941 Kolmogorov theory has had an overwhelming influence on theoretical and experimental investigations of the physics of turbulence. Major experimental efforts have been devoted to verifying Eq. (1); the conclusions have been mixed, but tend to its favor (see [2] for a recent account). Other predictions of the theory are, however, challenged by experimental data (see, e.g., [3]) and computer simulations [4,5], most notably the Reynolds number independence of the statistical distribution of fluctuations at scale $l_d \sim 1/k_d$. This has stimulated, over the last thirty years, a number of new proposals focusing on the "intermittency" of turbulence. Despite these efforts, a unifying fundamental framework has not yet emerged.

In the present paper, we focus on analyzing the Reynolds number dependence of turbulence statistics using data from direct numerical simulations of homogeneous isotropic Navier-Stokes turbulence over a range of R_λ. Isotropic turbulence has been simulated by integrating the Navier-Stokes equation with periodic boundary conditions, using the pseudospectral method [6]. Until recently, the resolution has been limited, by computer memory and speed, to 256^3 points in physical space, corresponding to a ratio of maximum to minimum wave number of 128. This permits good definition of the near-dissipation range of turbulent flows with $R_\lambda \leq 150$ [7,8]. We have performed simulations at resolution 512^3 on a Connection Machine CM-200 computer at the Advanced Computing Laboratory at Los Alamos National Laboratory. With this resolution and lower ones, we have been able to simulate stationary turbulent flows over a range of R_λ up to 200, with adequate resolution of the dissipation range. The result is a large database for the present analysis and for future studies.

In order to generate isotropic, periodic, stationary turbulent flow fields, a forcing term is added to the Navier-Stokes equation in wave-number space, only at the wave-number shells $k=1$ and 2. The forcing term is constructed to keep the total energy in each of the first two wave-number shells constant in time, a procedure first introduced in [9]. For a detailed description of the simulations, see [10]. Precisely the same forcing was used for most flows reported below, where the only parameter varied is the kinematic viscosity ν.

In this Letter, we examine the energy spectrum, a two-point statistic, and also the normalized higher moments (flatness factors) of the longitudinal velocity increment $\delta v_l = \mathbf{e}\cdot\mathbf{u}(\mathbf{x}+l\mathbf{e}) - \mathbf{e}\cdot\mathbf{u}(\mathbf{x})$ across a distance l in the direction $\mathbf{e}$. These quantities have been measured and reported by many authors for simulations of moderately-high-Reynolds-number flows, but their Reynolds number dependence has not yet been studied systematically for a substantial range of R_λ. The R_λ dependence of the derivative statistics was studied by Kerr [4] for a smaller range of Reynolds numbers than treated here.

Our main results are as follows: (1) Energy spectra rescaled by a characteristic dissipation wave number k_p (to be defined below) and the spectrum level at k_p collapse well into a single curve that also agrees accurately with experimentally measured spectra at much higher

Reynolds numbers; (2) the variation of k_p with R_λ is consistent with the Kolmogorov 1941 theory; (3) high-order flatness factors of velocity increments appear to be consistent with universal dependence on the separation distance, if the latter lies in the inertial range and is normalized by the forcing scale. The observed dependence implies intermittency growth with decrease of scale. We characterize this growth by a set of power-law exponents that are in agreement with previous numerical and experimental results.

In Fig. 1, we display a superposition of normalized isotropic energy spectra for $R_\lambda \approx 15$, 36, 100, 150, 160, 200. Wave numbers are normalized by k_p, the wave number at which the dissipation spectrum $k^2E(k)$ peaks, while $E(k)$ is normalized by $E(k_p)$. Minor adjustments have been made because k_p does not always lie exactly at a wave number of the discrete set permitted by the cyclic boundary conditions. The results plotted in Fig. 1 show that the normalized spectra at different Reynolds numbers collapse accurately to a single curve. This suggests the existence of a universal function $F(k/k_p)$, at least for the range of the Reynolds numbers explored here, such that the energy spectrum takes the form $E(k)=E(k_p) \times F(k/k_p)$. The higher-Reynolds-number spectra extend further to the left of k_p in the plot (smaller normalized wave numbers). Note that the spectrum at $R_\lambda = 150$ is obtained by using a time-independent forcing term similar to that used in [7]. The results indicate that the form of $F(k/k_p)$ is insensitive to the form of the forcing term.

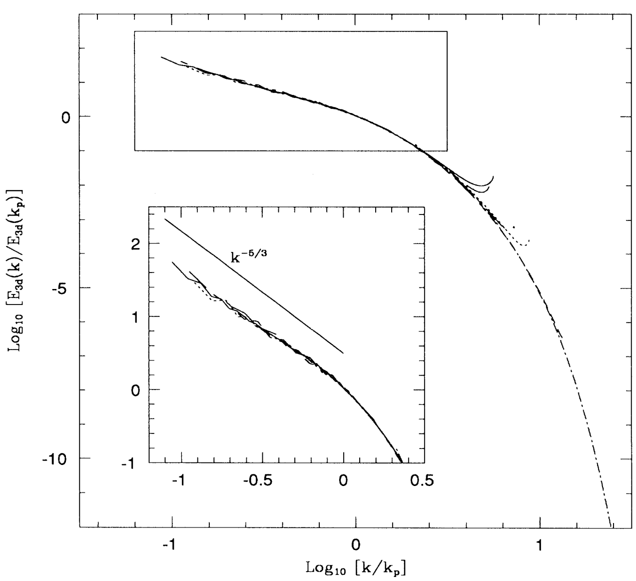

FIG. 1. Normalized isotropic energy spectra obtained from direct numerical simulations at various Reynolds numbers. Spectra are rescaled by the maximum dissipation wave number k_p and $E(k_p)$. Solid line: $R_\lambda \approx 200$; dotted line: $R_\lambda \approx 160$; dashed line: $R_\lambda \approx 100$; dot-dashed line: $R_\lambda \approx 36$; dot-long-dashed line: $R_\lambda \approx 15$; short-long-dashed line: $R_\lambda \approx 150$ obtained with a different (time-independent) forcing term. The inset zooms in on the area outlined by the rectangle.

These data support Kolmogorov's 1941 description [1] of turbulence energetics in two ways. The first is the evident collapse of the data in Fig. 1 in both the dissipation range and below ($k \le k_p$). The second is that the collapse at wave numbers $k/k_p \le 0.2$ [$\log_{10}(k/k_p) \le -0.7$] is to a curve that appears consistent with an asymptotic $k^{-5/3}$ inertial range. Further verification of the Kolmogorov 1941 description concerns the relation between k_p and R_λ, or $k_d = (\epsilon/\nu^3)^{1/4}$, the Kolmogorov dissipation wave number which can be independently measured using flow parameters ϵ and ν. Kolmogorov's prediction (1) implies a linear relation between k_p and k_d. In Fig. 2, we show a plot of k_p vs k_d. At large Reynolds numbers, k_p and k_d appear to have a linear relation. It should be stressed that the Kolmogorov relation (1) is not just an asymptotic law for high R_λ, and may also be valid for small R_λ, provided that k_d is replaced by k_p in (1). The above results indicate further that defining the rescaled wave number k_p in terms of the global flow parameters (ϵ and ν) does require that the Reynolds number be large. In summary, we conclude that the Kolmogorov 1941 description of turbulence energetics is supported in the high end of the range of Reynolds numbers explored here.

We do not attempt to determine the inertial-range exponent, because the available wave number range $k/k_p \le 0.2$ is quite narrow. It must be stressed that one cannot simply lay a straight edge on the data close to k_p and proclaim a power law with exponent equal to the slope of the line. It is essential to take into account the effects of the neighboring dissipation range. In order to obtain an inertial range extending more than a decade which is free of dissipation range effects, $k/k_p \le 0.02$ is needed, or $k_p \ge 50k_{\min}$, where $k_{\min}$ is the lowest wave number free of forcing. Taking $k_{\min}=3$ (meaning that forcing is applied only on the first two wave-number shells), this

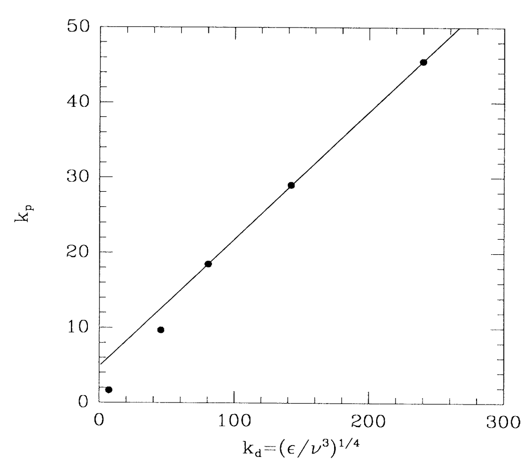

FIG. 2. The wave number of peak dissipation k_p vs the Kolmogorov dissipation wave number $k_d = (\epsilon/\nu^3)^{1/4}$. The straight line is drawn to highlight the linear relation between k_p and k_d at high Reynolds numbers.

estimate yields $k_p \approx 150$, which requires a resolution $\gtrsim 1500^3$; this computation should be possible on a teraflop computer.

A closer look at the form of $F(k/k_p)$ suggests that for wave numbers close to but below k_p, the energy spectra exhibit a somewhat flatter behavior than at lower k ($k/k_p \leq 0.2$). One possible explanation is that a flatter spectrum near k_p reflects some interesting dynamics related to vortex structures at scales near $1/k_p$, since these are scales where both the stretching and the diffusion of the vorticity are mostly important (k_p is the maximum wave number for the vorticity power spectrum). On the other hand, certain second-order closures [11] predict such flattening. These closures deal only with second-order statistics and cannot portray explicit vortex structures in physical space. They give a bump below k_p in a plot of $k^{5/3}E(k)$ because the spectrum falls rapidly when $k > k_p$. This rapid falloff effectively removes many triad interactions that contribute substantially to energy transfer at lower k. The result is inefficient energy transfer just below k_p and a consequent pile up of energy.

The question of the existence of universal energy spectra has been the subject of considerable discussion in experimental investigations. Recently, She and Jackson [2] did a study, similar to that described here, for a collection of experimental spectra with R_λ ranging from 130 to 13000, and found, after renormalization with respect to k_p, that all spectra do collapse quite well to a universal curve. In order to compare directly our numerical results with experimental spectra which are obtained from one-dimensional time series, we have computed one-dimensional energy spectra from our data set. The results are shown in Fig. 3 (lines), together with a few (normalized) experimental spectra (points) with R_λ up to 13000 [3,12]. It may be seen that our numerical spectra agree closely with experimental results at scales close to the dissipation cutoff k_p, even though the Reynolds numbers are far apart. This comparison demonstrates that both experimental and numerical energy spectra exhibit the same universal features. In addition, the comparison supports the reliability of the simulations at large k.

Questions still exist concerning the behavior of the energy spectra at much larger Reynolds numbers. The above results by no means imply universality to infinite R_λ, both because of the limited range of the Reynolds number we have explored and because of resolution limitations for $k \gg k_p$. The present results do suggest that a simple kind of universality holds to a good approximation over a wide range of Reynolds numbers.

Next, we examine high-order statistics of velocity increments. Second-order quantities such as the energy spectra give very limited information about turbulent structures. Coherent vortex structures in the form of ribbons and filaments have been observed in both numerical simulations and laboratory experiments [13]. They constitute an important aspect of turbulence dynamics. It has been suggested [14] that these structures may be particularly relevant to the growth of intermittency with R_λ. However, it is still not clear to what extent intermittency effects are universal.

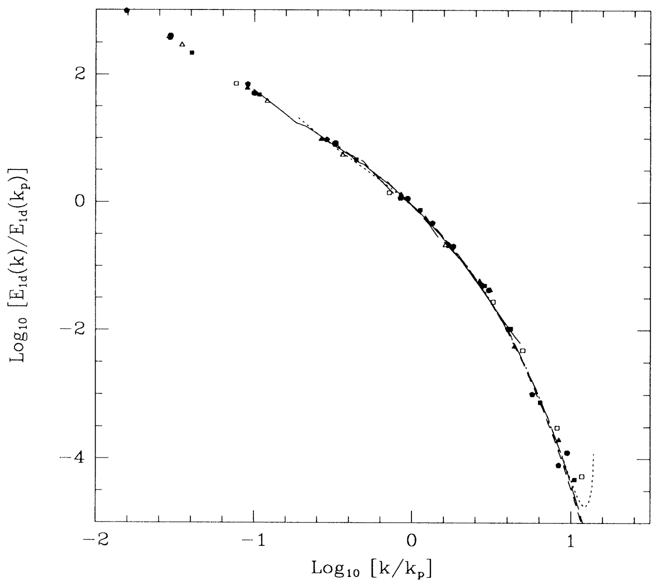

FIG. 3. Normalized 1D energy spectra obtained from direct numerical simulations at various Reynolds numbers. Spectra are rescaled according to the maximum dissipation wave number k_p and its energy $E(k_p)$. Solid line: $R_\lambda \approx 200$; dotted line: $R_\lambda \approx 160$; dashed line: $R_\lambda \approx 100$; dot-dashed line: $R_\lambda \approx 70$. Points are a collection of experimental data [3,12].

Previous numerical studies [7,10,14] and a number of experimental studies (e.g., [3]) have shown that velocity differences become increasingly intermittent and non-Gaussian for decreasing separation distances. The departure from Gaussian behavior is manifested by the increasing flatness factors. In order to investigate possible universal features in the growth of flatness factors, we have computed [10] the spatially averaged probability density functions (PDFs) of the velocity increments for a few separation distances and for several Reynolds numbers; from these PDFs we evaluate the flatness factors. We observe that these spatial flatness factors fluctuate in time, although we have a fairly large spatial sample size ($\sim 512^3 \approx 1.3\times10^8$). The level of fluctuations increases as separation distance decreases (smaller scale). At large scales (inertial-range distances), this fluctuation is negligible (about 1%), while at smallest dissipation length scale (across two mesh points), the fluctuation level is about 5%–10%. We have also consistently observed increasing fluctuations of the spatial-averaged velocity derivative flatness with increasing Reynolds numbers. These observations indicate that small scales are more intermittent in time than large scales.

In Fig. 4, the fourth-order flatness factor $\langle(\delta v_r)^4\rangle/\langle(\delta v_r)^2\rangle^2$ is shown (in log-log coordinates) as a function of the separation distance r for simulations at several

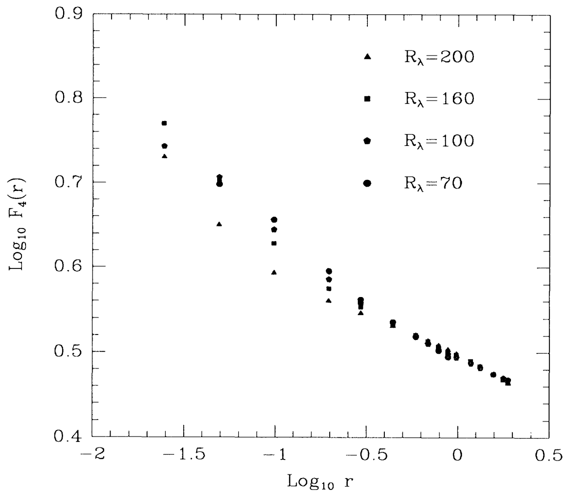

FIG. 4. Flatness factors of the longitudinal velocity differences vs separation distances at several Reynolds numbers.

Reynolds numbers. Remarkably, at large scales the flatness data for different R_λ superpose very closely (skewness data do not superpose, a result not shown here). They grow algebraically as r decreases, and then grow faster for r smaller than a certain distance r_d. At larger Reynolds numbers, the algebraic range extends further (r_d is smaller). Here we have evidence of an intermittency growth associated with a range which becomes more extended as R_λ increases. This can probably be described as inertial-range intermittency growth rather than the stronger intermittency associated with smaller dissipation scales. Two distinct kinds of intermittency growths already have been observed in numerical simulations in [15].

What is in common in all our simulations is the forcing which always acts at $k=1$ and 2. Thus, we can also consider the separation distance normalized by the scale of the forcing where the cascade of energy to small scales starts. The universal dependence of the flatness factors on separation distance indicates that the departure from Gaussian behavior at a length scale r only depends on r/r_0 where r_0 is the size of the energy containing eddies. In other words, intermittency grows with each step of the inertial-range energy cascade.

A quantitative characterization of the inertial-range intermittency growth is the best-fit exponent α of the algebraic dependence: $F_4(r) \sim r^\alpha$. A least squares fit in the range $0.4 \leq r \leq 1.9$ yields $\alpha = -0.105 \pm 0.01$, where the error bar specifies the range of scattering of the exponent at different Reynolds numbers. This value is close to earlier numerical results -0.11 [7] and experimental results -0.09 [3]. The Kolmogorov 1941 theory predicts that $\alpha=0$. A nonzero value of α, if it persists to high Reynolds numbers, invalidates Kolmogorov's 1941 description of high-order velocity structure functions. It remains unanswered why Kolmogorov's description of turbulence energetics is so accurate. One plausible explanation [5] is that the structures that dominate intermittency and higher statistics do not play a significant role in the mean transfer of energy. The latter may be mediated principally by flow regions where intermittency is not strong.

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