Self-Affinity of Multiplicity Fluctuation in the Phase Space of Multiparticle Production

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It is argued that, accounting for the anisotropy of phase space, the local fluctuations of multiplicity in high energy multiparticle production are self-affine rather than self-similar. Evidence is given confirming self-affinity in phase-space distributions. A method is proposed to extract the characteristic parameter of self-affinity, the Hurst exponent, from the experimental data iteratively.

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The study of multiplicity fluctuations in decreasing phase-space intervals in high energy multiparticle production using the method of factorial moments was proposed a few years ago [1]. It is expected that fluctuations of nonstatistical origin ("dynamical fluctuation" or "intermittency") may be present and the distribution in phase space may possess a fractal structure [2]. Until now, it has generally been believed that these effects should be observed in a higher-dimensional analysis [3] instead of a one-dimensional (rapidity) analysis. However, as is well known, the phase space in multiparticle production is anisotropic, as indicated by the name "longitudinal phase space" first introduced by Van Hove [4] in 1969. Thus a basic question is: What is the influence of the anisotropy of phase space on nonstatistical fluctuations? In this Letter this problem will be discussed in some detail.

Recently, fluctuations in azimuthal (ϕ) distributions have aroused some interest [5] in connection with the possible existence of jetlike (tower) or ringlike (wall) structures. However, a more careful study [6] shows that such structures are mainly due to statistical fluctuations, having no essential significance. For simplicity, it is reasonable to neglect the azimuthal distribution first and study the multiplicity fluctuations in two-dimensional phase space $(p_{\parallel},p_{\perp})$ or $(y_{\parallel},\ln p_{\perp})$ or the corresponding transformed variables [7]. In a two-dimensional analysis the variables chosen in different directions should be consistent with each other. There are reasons to believe that the variables $(y_{\parallel},\ln p_{\perp})$ are more relevant [2]. In this Letter we will concentrate on the formulation of a method for treating the anisotropy of phase-space fluctuations and leave the problem of variable choice for further study. Therefore, we will in the following use $x_{\parallel},x_{\perp}$ to denote the variables adopted in the two phase-space directions.

The usual procedure for analyzing higher-dimensional, in particular two-dimensional, intermittency is to divide the corresponding phase space subsequently into subcells by shrinking equally in each dimension. This, in the language of fractal geometry, corresponds to self-similar fractal structure [8]. However, the phase space in high energy multiparticle production is anisotropic. The longitudinal momenta $p_{\parallel}$ are usually large, bounded only by energy-momentum conservation, while the transverse ones $p_{\perp}$ are limited to small values with an average of 0.3–0.4 GeV/c by a production mechanism which is in some way still unknown at present. For this reason, we cannot simply expect the fluctuations or scaling properties to be the same in both directions. It is more reasonable to assume that the scaling behaviors in the longitudinal and transverse directions are different.

Our present case is very similar to the three-dimensional space of landscape, where the vertical direction is a special one due to the existence of gravity, which causes the vertical (or altitude) variations of landscapes to be scaled differently from the horizontal ones. According to Mandelbrot [8], when some given patterns are scaled differently in different directions (i.e., scaled anisotropically), they are called self-affine fractals. Thus, the anomalous scaling of landscape is self-affine in the vertical (z, x or z,y) plane, and is self-similar in the horizontal (x,y) plane. In this terminology, the phase-space structure in high energy multiparticle production should be self-affine in the plane consisting of the longitudinal and transverse directions while self-similar in the transverse plane. In the following we will concentrate our discussion on the evidence of self-affine property between the anomalous scaling of longitudinal and transverse directions and on the method for its experimental observation.

We choose the self-affine function for multiparticle production as

$$C_q(\delta x_a,\delta x_b)=\sum_{i,j}\langle p_{i,j}^q\rangle\Big/\sum_{i,j}\langle p_{i,j}\rangle^q\,, \tag{1}$$

where $p_{i,j}$ is the probability for a particle to fall into the i,jth cell in (x_a,x_b) space. The subscripts a and b are used to denote the two different phase-space directions —longitudinal and transverse. $C_q(\delta x_a,\delta x_b)$ describes the dynamical fluctuations of particle distribution in two-dimensional phase space. It can be obtained from the experimental data by the method of factorial moments [1].

The self-affine transformation is defined as $\delta x_a \to \delta x_a/\lambda_a$; $\delta x_b \to \delta x_b/\lambda_b$, with $\lambda_a \le \lambda_b$. The shrinking ratios in this transformation are characterized by a parameter

$$H=\ln\lambda_a/\ln\lambda_b \quad (0\le H\le 1)\,, \tag{2}$$

called roughness or the Hurst exponent [8]. If and only if the shrinking ratios in the two directions satisfy Eq. (2)

with a particular H value, the function $C_q(\delta x_a,\delta x_b)$ will have a well defined scaling property:

$$C_q(\delta x_a,\delta x_b)=\lambda_a^{\varphi_q^{(a)}}\lambda_b^{\varphi_q^{(b)}}C_q(\lambda_a\delta x_a,\lambda_b\delta x_b)\,. \tag{3}$$

In this sense, it is the Hurst exponent H that characterizes the self-affine property of $C_q(\delta x_a,\delta x_b)$.

Let us examine two special cases: $H=0$ and $H=1$.

(1) For $H=0$, $\lambda_a=1$, $\lambda_b=\lambda\neq 1$, which means that δx_a is constant, the function $c_Q(\delta x_a,\delta x_b)$ depends only on one variable δx_b. In this case, the scaling property exists in one dimension $C_q(\delta x_b)=\lambda^{\varphi_q}C_q(\lambda\,\delta x_b)$, and $\ln C_q$ vs $\ln M_b$ ($M_b=\lambda_b^\nu$ is the total number of subintervals) should strictly be a straight line. We already know from experimental data [2] that this is not true; cf. Fig. 1(b); So H is not zero.

(2) For $H=1$, $\lambda_a=\lambda_b=\lambda$, self-affine reduces to self-similar. If this is the case, $\ln C_q$ vs $\ln M$ ($M=\lambda^{2\nu}$ is the total number of subcells) should be a straight line in a two-dimensional analysis with isotropic shrinkage of phase space. It seems that the experimental data [2] do not support this possibility either. The present available data on the $\ln C_q$ vs $\ln M$ curves in two- as well as three-dimensional analysis are in general bending upward; cf. Fig. 1(b). (The y-$p_\perp$ curve in the figure is bending upward after $\ln M=2$.) So H is not equal to unity either.

Since both one- and two-dimensional self-similar evolution are not good candidates for a description of multiparticle production, the next possibility is two-dimensional self-affinity. In order to check this, we have to consider what phenomena are expected to be observed in the self-affine case and how to extract the Hurst exponent, characterizing the self-affine property, experimentally.

In order to get some hints for answering these questions, we have constructed a two-dimensional self-affine random cascading model. First, take a two-dimensional phase-space region $\Delta x_a\Delta x_b$ and divide it subsequently with the subdividing ratios λ_a and λ_b equal to two integers, respectively. The probability for a subdivision is taken to be

$$w_{ij}=\frac{1+\alpha r_{ij}}{\sum_{i=1}^{\lambda_a}\sum_{j=1}^{\lambda_b}0001(1+\alpha r_{ij})}\,, \tag{4}$$

where α is the parameter of fluctuation strength, $0\le\alpha\le 1$; r_{ij} is a random number in the interval $[-1,1]$. The w_{ij} as defined in Eq. (4) have the following properties: (1) The probability is normalized to 1 in each subdivision; (2) the possible values of w_{ij} are symmetric with respect to different values of i,j; (3) the smallest and largest possible values of w_{ij} are 0 and 1, respectively. After ν steps, the probability in a subcell is $p_{i_\nu j_\nu}^{(\nu)}=w_{i_1j_1}^{(1)}w_{i_2j_2}^{(2)}\cdots w_{i_\nu j_\nu}^{(\nu)}$, where $i_\nu=1,\ldots,\lambda_a^\nu, j_\nu=1,\ldots,\lambda_b^\nu$.

Using this model, we have made a Monte Carlo (MC) simulation for a self-affine cascading process up to $\nu=8$. For simplicity, we have taken $\lambda_a=2$, $\lambda_b=3$, $\alpha=0.3$. The data sample thus obtained is then used to calculate the various moments. The results of one-dimensional projection in both the a and the b directions are shown in Fig. 1(a). The two curves are distinct, reflecting the anisotropy of the two-dimensional space in consideration. In Fig. 1(b) are shown the experimental data from NA22 on the factorial moments in longitudinal (y) and transverse (p_T) directions for comparison. Using our MC sample we have also calculated the two-dimensional moments by isotropic shrinkage of the space (self-similar analysis, i.e., self-affine analysis with $H=1$). The result is shown in Fig. 2 as a solid line. It is not straight but is bending upward, giving more evidence for the self-affinity with $H\neq 1$ of the sample in consideration. The result for a self-affine analysis (anisotropic shrinkage of the space) with the Hurst exponent $H=\ln 2/\ln 3$ exactly the same as in the model is also shown as a dashed line in Fig. 2. It is

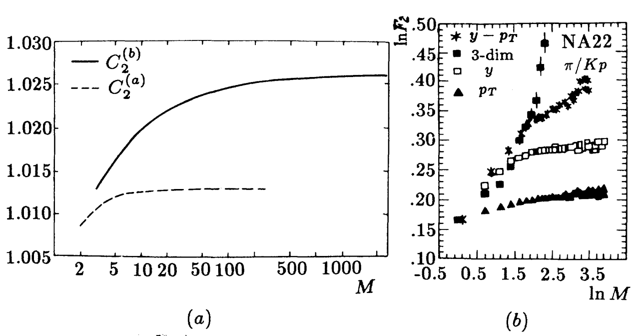

FIG. 1. (a) The calculated results for the one-dimensional moments $C_2^{(a)},C_2^{(b)}$ as functions of the number M of subintervals. (b) Na22 results of one-dimensional factorial moments F_2 in longitudinal y and transverse p_T directions and of two- and three-dimensional factorial moments with isotropic shrinkage of phase space. Data are taken from Ref. [12].

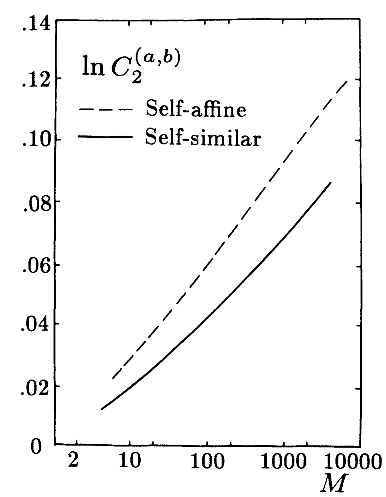

FIG. 2. The calculated results for the logarithm of two-dimensional moments $C_2^{(a,b)}$ as function of the number M of subcells.

straight of course.

Note that the inequality of the moments $C_q^{(a)}, C_q^{(b)}$ (or the factorial moments $F_q^{(a)}, F_q^{(b)}$) is not the main point. If the fluctuations in the two directions were completely independent, the distinction between $C_q^{(a)}$ and $C_q^{(b)}$ ($F_q^{(a)}$ and $F_q^{(b)}$) would be a trivial consequence of the anisotropy of phase space. The point is that the fluctuations in the whole phase space are correlated as indicated by the curvature of the log factorial moments versus $\ln M$ in one dimension. How to describe the correlated anisotropic fluctuation is not trivial. It is deeply connected with the relative evolution of particle production in different directions which results in a self-affine fractal structure in the multiplicity fluctuation in phase space. The Hurst exponent H is the appropriate parameter, describing the self-affinity of the correlated anisotropic fluctuation quantitatively.

A very important question is how to extract the Hurst exponent from experimental data. In order to answer this question, let us write down the formulas for the one-dimensional projections of C_2 [9]:

$$C_2^{(a)}(\delta x_a) = a_2\left[1 - b_2\left(\frac{C_w^2}{\lambda_b}\right)^{\nu}\right],$$
$$C_2^{(b)}(\delta x_b) = a_2'\left[1 - b_2'\left(\frac{C_w^2}{\lambda_a}\right)^{\nu}\right], \quad (5)$$

where $C_w^2 = \langle w^2\rangle/\langle w\rangle^2$ and the coefficients b_2 and b_2' are greater than zero. Since $(C_w^2/\lambda) < 1$, both $C_2^{(a)}(\delta x_a)$ and $C_2^{(b)}(\delta x_b)$ tend to saturate for large ν. The saturated values are $C_{2\max}^{(a)} = a_2$, $C_{2\max}^{(b)} = a_2'$, respectively. Defining

$$R(\nu) = \frac{C_{2\max}^{(b)} - C_2^{(b)}(\delta x_b)}{C_{2\max}^{(a)} - C_2^{(a)}(\delta x_a)}, \quad (6)$$

we see that

$$\ln R = \ln(a_2' b_2'/a_2 b_2) + (H^{-1} - 1)\ln M_a, \quad (7)$$

where $M_a = \lambda_a^{\nu}$ is the number of subintervals in the a direction. In deriving Eq. (7) the relation (2) has been used. Thus, from the slope of the line $\ln R$ vs $\ln M_a$ the Hurst exponent H can be obtained. Note that Eqs. (6) and (7) are based on Eq. (5), which is valid independent of the concrete distribution of w and the exact values of λ. Therefore the Hurst exponent H obtained in this way is largely model independent. The only requirement is that in the phase-space structure there is fractality produced by an α-model-style cascade of independent variables.

The above method, however, cannot be used directly in the analysis of experimental data. The reason is that when deriving Eqs. (6) and (7) the values of the "number of steps" ν in both Eqs. (5) are assumed to be the same, which is not straightforward to realize in experimental data analysis. In order to solve this problem we propose a self-consistent iteration method.

Rewrite Eqs. (2), (5), (6), and (7) as

$$\lambda_b = \lambda_a^{1/H}; \quad C_2^{(a)}(M_a) = a_2[1 - b_2(C_w^2)^{\nu}/M_a^{1/H}];$$
$$C_2^{(b)}(M_b) = C_2^{(b)}(M_a^{1/H}) = a_2'[1 - b_2'(C_w^2)^{\nu}/M_a],$$
$$R(M_a) = \frac{C_{2\max}^{(b)} - C_2^{(b)}(M_a^{1/H})}{C_{2\max}^{(a)} - C_2^{(a)}(M_a)}, \quad (8)$$
$$H^{-1} = 1 + d\ln R(M_a)/d\ln M_a, \quad (9)$$

respectively. In the zeroth approximation let $H = H^{(0)} = 1$ and put it into Eq. (8) to get $R(M_a)$. Then calculate Eq. (9) at some fixed (unsaturated) value of M_a; take the result $H^{(1)}$ as the first approximation of the Hurst exponent H; insert it again into Eq. (8), and calculate (9) to get $H^{(2)}$;.... This procedure continues until two neighboring values of H, say, $H^{(i+1)}$ and $H^{(i)}$, are close enough. Note that, with this value of H, $\ln R(M_a)$ vs $\ln M_a$ will be a straight line and the particular value chosen for M_a becomes irrelevant. In Fig. 3 are shown the H values obtained in each step. It converges well to the theoretical value $H = \ln 2/\ln 3 = 0.631$.

The above model is a toy model. The real multiparticle production processes are much more complicated. However, if the phase space in multiparticle production is really self-affine in the above-mentioned sense and the phase-space variables have been chosen correctly, then the proposed iteration method should be applicable. Therefore, we would suggest doing the following in experimental data analysis:

First, calculate the factorial moments $F_2^{(a)}, F_2^{(b)}$ in longitudinal and transverse directions and plot them as functions of $\ln M$.

Next, check whether the two curves obtained are distinct and whether they both become saturated at large M when the experimental resolution in phase space is high enough. If so, call the one saturated quicker as $F_2^{(a)}$, the other one as $F_2^{(b)}$, and do the iteration procedure (8) and (9). If the result converges to a certain value H, take it as a characteristic parameter of self-affinity in the production process in consideration.

The anisotropy in phase space, which has been ob-

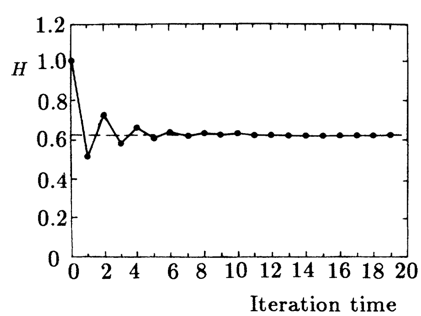

FIG. 3. Calculated values (circles) of the Hurst exponent H as a function of iteration time. The dashed line is the theoretical value $H = 0.631$.

served since the sixties, is deeply connected with the mechanism of particle production. Up to now it has been examined only with respect to the average momentum distributions. In this Letter we have discussed its influence on the local fluctuations. It is argued that the corresponding phase space should be scaled differently in different directions, so that the phase-space structure is self-affine. The distinction between the factorial moments in longitudinal and transverse directions and the nonlinearity of the logarithm of the two-dimensional factorial moments with isotropic shrinkage of phase space versus $\ln M$ are evidence for self-affinity. An iterative method for extracting the characteristic parameter of self-affinity, the Hurst exponent, from experimental data is proposed based on the random-cascading α model. If the iteration procedure really converges, it will be another support for the self-affinity of phase space in the above-mentioned sense. The Hurst exponent thus obtained will be an important parameter for describing the production process in consideration.

Recently, the scaling of moments in the Q^2 variable has been observed [10]. It is probably due to the geometrical fluctuations of interaction volume [11], while the scaling of factorial moments in momentum variables studied in this Letter is connected with the nonstatistical fluctuation in phase space [1]. Evidently, in the study of both kinds of scaling the anisotropy of phase as well as geometrical space should be taken into account. To extend the approach proposed in this Letter to the case of interaction volume fluctuation is worthwhile. Research in this direction is in progress.

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