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Solitons on Oscillating and Rotating Backgrounds

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We show that kink solitons may exist on top of a rotating background as *localized objects* if the background dynamics is phase locked to a rapidly oscillating ac force. The result is demonstrated analytically for the sine-Gordon model, showing excellent agreement with numerical simulations. In the context of long Josephson junctions this phenomenon means that the well known Shapiro steps in the current-voltage characteristics change into a set of the zero-field steps according to the number of propagating fluxons.

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As is well established in the literature, soliton bearing models are very important in describing nonlinear dynamics of many physical systems in the one- and higher-dimensional approximation. In order to describe interaction with the surroundings, these models are often perturbed by external forces, dissipative losses, as well as other types of perturbations (see, e.g., Ref. [1]). Typically, these perturbations are considered to be relatively small, and hence, the integrable model, having exact soliton solutions, becomes nearly integrable, which implies that the solitons maintain their essential characteristics from the unperturbed system [2, 3]. This has been successfully demonstrated for different types of solitons under the influence of weak perturbations. It has been shown that applying an ac force to a damped envelope soliton system may in fact compensate for the dissipation and hereby maintain the soliton as a stable object [2, 3]. For topological solitons (kinks), small external forces (ac or dc) give rise to stationary propagation in a damped system [3–6]. Relatively small perturbations of the system have also proven to cause more complicated dynamics involving coexisting states of bunched kinks and nontrivial background states [7]. But what happens with solitons in *strongly perturbed* systems? Large amplitude forces will have a dramatic influence and the nonlinear system becomes highly nonintegrable showing decay of topological solitons as localized objects [8].

The purpose of this Letter is to demonstrate, analytically and numerically, that very large perturbations of a system do not necessarily prevent the existence of solitons. We show that the solitons may have a predominant role as elementary nonlinear excitations, if the system dynamics can be described by different time scales characterizing slowly and rapidly changing values. We have chosen the sine-Gordon (SG) model as a particular, but rather general, example with numerous applications. We show that a large amplitude, high frequency force may phase lock the sine-Gordon field in an oscillating or rotating state and hereby create a mechanism (an effective gravitation field) for supporting solitons (note that for a high frequency driven system without spatial extension this is analogous to the Kapitza problem of the

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pivot modulated pendulum [9]). The present study is physically relevant for, e.g., long, but finite size, one-dimensional Josephson junctions driven by a large amplitude, high frequency bias current. The main physical consequence of our results is that the existence of localized 2π kinks (magnetic fluxons) gives rise to the so-called zero-field step (ZFS) like singularities around the well known Shapiro steps [10] in the current-voltage (IV) characteristics of the system with no spatial modulation.

We consider the perturbed SG equation given by [3]

$$\phi_{xx} - \phi_{tt} - \sin\phi \;=\; \alpha\phi_t - \eta - \epsilon\sin\Omega t\,. \tag{1}$$

For a long Josephson junction, ϕ describes the quantum mechanical phase difference between the two superconductors, defining the junction. The space dimension x is normalized to the Josephson length, and the time dimension t is normalized to the inverse plasma frequency of the junction. Tunneling of quasiparticles through the junction is described by the dissipative term $\sim\alpha$; the bias current density (normalized to the critical current density) forced through the junction is given by the dc term η and the ac term $\sim\epsilon$, where Ω is the normalized frequency. Let us now assume that the system is of finite length L so that the boundary conditions are periodic,

$$\phi(x=0,t) \;=\; 2\pi n + \phi(x=L,t)\,, \tag{2}$$

where n is an integer, denoting the number of trapped 2π-kink solitons. For these types of boundary conditions the dispersion relation of the linear modes are described by the discrete dispersion relation,

$$\omega_l^2 \;=\; \left(\frac{2\pi}{L}l\right)^2 + 1 \quad (l=0,1,2,\ldots)\,. \tag{3}$$

From this we find that as long as the driving frequency is not too close to one of the resonances, ω_l, we can avoid the spatially modulated linear modes. Similar arguments can be made for open boundary conditions, but for simplicity we have chosen here to consider the periodic boundary conditions only. Note that if the system is of infinite length the only way of avoiding the linear modes is to assume that the system is discrete [11]. Driving the system, Eq. (1), with a high frequency ac force, we will separate the field into two time scales,

$$\phi \;=\; \xi + \chi t + \psi\,, \tag{4}$$

where ξ is the short time-scale part, oscillating with the frequency Ω, ψ is the long time-scale part, and χ is the average frequency of rotation for the background field. In this analysis we will keep only the fundamental frequency Ω of the short time-scale part,

$$\xi \;=\; A\sin\Xi\,,\;\; \Xi = \Omega t + \nu\,, \tag{5}$$

where ν is a constant phase. Inserting Eqs. (4) and (5) into Eq. (1) and assuming that the background field is phase locked to the external signal, $\chi = \pm k\Omega$, k being an integer, we get the following equations for the separated time scales:

$$\psi_{xx} - \psi_{tt} - J_k(A)\sin(\psi - 2k\nu) \;=\; \alpha k\Omega - \eta + \alpha\psi_t\,, \tag{6}$$

$$A\Omega^2\sin\Xi \;=\; \alpha A\Omega\cos\Xi - \epsilon\sin\Omega t\,, \tag{7}$$

where the latter equation is derived for $\Omega^2 \gg 1$. The amplitude A of the fast oscillation is then obtained from Eq. (7) to be

$$A \;=\; \frac{\epsilon}{\Omega\sqrt{\Omega^2+\alpha^2}}\,. \tag{8}$$

Making the substitution, $\Psi = \psi - 2k\nu$, it is clear that the "slow" field Ψ is described by the SG equation (6), where the effective gravitation field $J_k(A)$ (the Bessel function of first kind, kth order) is determined by Eq. (8). The dynamics of the field are then completely determined by the separation into the different time scales. We note that, since the ψ field obeys a simple perturbed SG equation, we can apply all the results obtained in the literature for SG dynamics directly on Eq. (6). It is important to note that Eq. (6) is only valid if the short time-scale field is phase locked, since this is the mechanism determining the effective gravitation field. Hence, Eq. (6) is valid only for bias current densities η in the intervals

$$k\Omega\alpha - |J_k(A)| \;<\; \eta \;<\; k\Omega\alpha + |J_k(A)|\,. \tag{9}$$

As an example of how well Eq. (6) describes the dynamics of the strongly driven SG system in an oscillating or rotating mode, we will study kink-soliton motion. In order to study the steady state dynamics we will apply the periodic boundary conditions, Eq. (2), on a finite size system. The simplest possible analysis of a dc driven kink is to look at the constant velocity properties. The kink wave solution to the unperturbed system [left hand side of Eq. (6)] is given by

$$\Psi \;=\; 2\,\sigma\,\mathrm{am}(\sqrt{|J_k(A)|}\gamma(W)(x-X)q^{-1}) + \frac{\pi}{2}(1+\varsigma)\,, \tag{10}$$

where $\sigma = \pm 1$ is the polarity of the kink, X is its position, $\gamma(W) = (1-W^2)^{-1/2}$ is the inverse Lorentz contraction, $W = \dot{X}$ being the velocity of the traveling wave, and $\varsigma = \pm 1$ denotes if the ground state is 0 or π. Here am denotes Jacobi's elliptic amplitude depending on the elliptic modulus q $(0 < q < 1)$, where the modulus must obey the condition

$$q\,K(q) = \frac{L\sqrt{|J_k(A)|}}{2n}\gamma(W). \tag{11}$$

The wave equation (10) describes a propagating sequence of n localized kinks. As a measure of the localization we define the compression κ as

$$\kappa \;\equiv\; |\Psi_x|_{\max} \;=\; 2\gamma(W)q^{-1}\sqrt{|J_k(A)|}\,. \tag{12}$$

From Eq. (6) we easily find the steady state velocity of the soliton(s) to be given by

$$W_k = \sigma \frac{\pi(\eta - \alpha\kappa\Omega)L}{8\alpha n} \frac{1}{K(q)E(q)}, \tag{13}$$

where E is the complete elliptic integral of the second kind; A is given by Eq. (8).

The IV curve of the annular junction, operated so that the background field ξ is phase locked (on a Shapiro step), is then given by the contribution from the rotating field and the soliton field, respectively,

$$V_k = \left\langle \frac{1}{L} \int_0^L \phi_t dx \right\rangle = k\Omega + |n| \frac{2\pi}{L} \sigma W_k, \tag{14}$$

where $\langle \cdots \rangle$ means the temporal average, V_k being the normalized voltage of the kth step and n the number of kinks in the system. Clearly, this expression shows that there are no *vertical* Shapiro steps in the IV characteristics of a spatially extended junction if fluxons are trapped inside. Going through the kth step, varying the bias current in the interval defined in Eq. (9), the variation in the voltage will be approximately in the interval $k\Omega - |n|2\pi/L < V_k < k\Omega + |n|2\pi/L$.

We have performed a series of numerical simulations of the strongly driven system, Eq. (1), with the periodic boundary conditions defined in Eq. (2). As mentioned above the choice of those conditions has been made out of pure convenience for the simulations, since the number of trapped kinks is predefined and maintained during each simulation. We note that the open boundary conditions can support localized kinks in the oscillating and rotating systems as well, but the number of solitons is not predefined and we have therefore avoided these boundary conditions.

We have investigated many different parameter sets and they all show excellent agreement with the analytical predictions as long as the basic assumptions are fulfilled: $\Omega^2 \gg 1$ and $\Omega \neq \omega_l$. We will here show the results for the parameters $L = 10$, $\alpha = 0.1$, and $\Omega = 12.9$, where the frequency of the driving force has been chosen to lie in a gap, $\omega_{20} \simeq 12.6 < \Omega = 12.9 < 13.2 \simeq \omega_{21}$, of the linear resonances [see Eq. (3)].

Figure 1 shows the fields ϕ and ϕ_x as a function of space and time for $n = 1$, $\eta = 2.7$ ($k = 2$), and $\epsilon = 500$ over 10 periods of the driving field. As is evident from Fig. 1 we find a *localized* kink propagating in the positive direction ($\eta - \alpha\Omega k > 0$) with constant velocity. The background field is seen to be phase locked with an average velocity of 2Ω in accordance with the bias point at the second ($k = 2$) Shapiro step. Note that the fast temporal modulation of the background field does not seem to affect the spatial modulation defining the kink [see Fig. 1(b)]. Similar types of the dynamics have been observed for more than one kink in the junction ($n = 2, 3, \ldots$). The kinks are localized and travel in a steady state with a constant velocity on the rotating background.

Figure 2 shows normalized IV curves for $n = 0, 1$. The case $n = 0$ shows the same characteristics as of a small Josephson junction under the influence of an ac force. The vertical Shapiro steps ($k = 0, 1, 2, 3, 4$) are visible and their heights, $2|J_k(A)|$, are consistent with the theory. For $n > 0$ we find, in accordance with Eq. (14), that the steps are not vertical due to the propagating soliton(s). Note that the ground state for $k = 0$ is $\Psi = \pi$ [$J_0(A) < 0$]. This peculiarity has no physical importance for the Josephson junction since the phase difference ϕ is not directly observable. Looking at the insets in Fig. 2, where parts of the IV curves are shown in detail (for $k = 1$ and $k = 3$), we see that the effective equation (6) does in fact describe the field *very* well. Here, the dotted curves show the perturbation result, Eq. (14), and the solid lines show the numerically obtained IV curves. The agreement is so good that the results of the perturbation treatment are almost identical to the results of the numerical simulations.

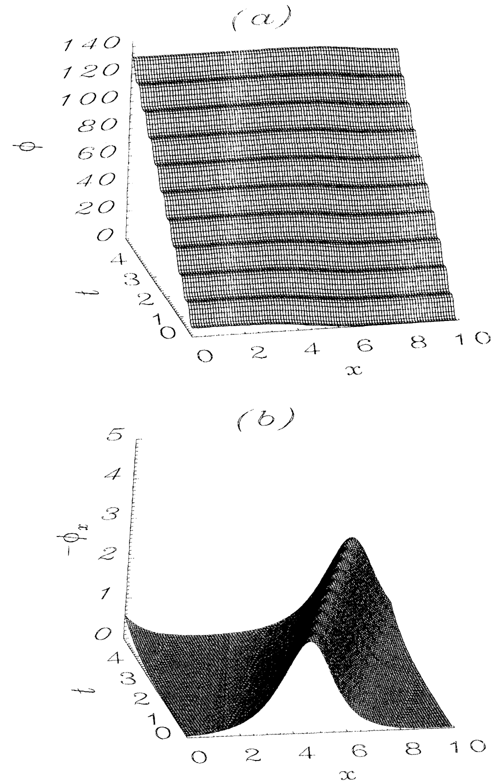

FIG. 1. The steady state profile of the ϕ (a) and ϕ_x (b) fields as a function of space over 10 periods of the external drive. Parameters are $L = 10$, $\alpha = 0.1$, $\epsilon = 500$, $\Omega = 12.9$, and $\eta = 2.7$ ($k = 2$). The boundary conditions are those of Eq. (2) for $n = 1$ ($\sigma = 1$).

In Fig. 3 we have shown the compression κ defined in Eq. (12) together with the results of numerical simulations (markers). Again we find excellent agreement between the theory and the simulation. When the background field is not phase locked to the ac force, the localization effect is absent and the kink disappears as a

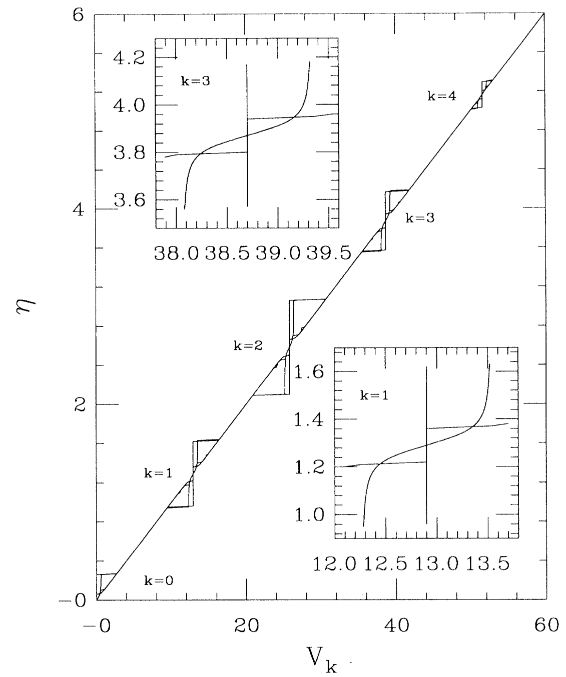

FIG. 2. The normalized IV curves for the system with the parameters $\alpha = 0.1$, $L = 10$, $\epsilon = 500$, $\Omega = 12.9$, and $n = 0, 1$ (the case $n = 0$ is the vertical Shapiro step, and $n = 1$ is the zero-field step like curve). The insets show details of the curves for $k = 1$ and $k = 3$. Note that there are both solid [the results of numerical simulations on Eq. (1)] and dotted curves [the results of the analytical treatment, Eq. (14)]. Note that the agreement between numerical and analytical results is close to perfect—the dotted and solid curves are almost overlapping.

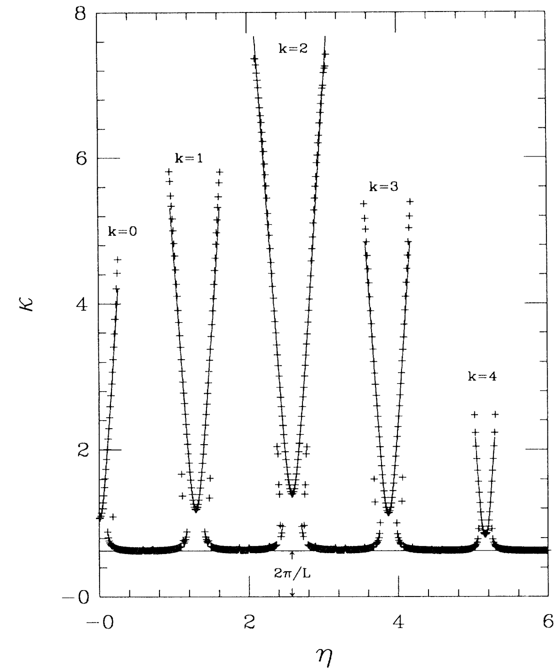

FIG. 3. The compression κ as defined in Eq. (12). System parameters are as in Fig. 2. The solid curves are the analytical results and the markers are the corresponding results of numerical simulations.

localized object. This feature is visible in Fig. 3, where we have drawn a horizontal line $\kappa = 2\pi/L$ denoting the compression of the unlocalized state.

In conclusion, we have demonstrated that a sine-Gordon system with a rotating background may support localized kink solitons if the background dynamics are phase locked to a rapidly oscillating ac force. This result, predicted by analytical arguments based on the method of separating the dynamics in long and short time scales, shows that the localization of the kink is related to the characteristic phase of synchronization and, thus, the kink dynamics in the long time scale depend strongly on the parameters in the short time-scale regime. The analytical results are confirmed by direct numerical simulations, showing excellent agreement with the theoretical predictions. For the theory of the Josephson junctions the above results mean that the well known vertical Shapiro steps in the IV curve of a junction coupled to an external ac force imply a localization effect, which makes the system able to support stable propagation of localized fluxons. This propagation affects the IV curves so that the usual Shapiro steps change into zero-field steps according to the number of propagating fluxons.

We note finally that long Josephson junctions are well within current fabrication capabilities. Experiments, verifying the results in this paper, should therefore be easy to perform.

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