Topology Transitions and Singularities in Viscous Flows

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Topological reconfigurations of the boundaries of thin fluid layers in Hele-Shaw flow are studied. A systematic treatment of the dynamics of the bounding interfaces is developed through an expansion in the aspect ratio of the layer, yielding nonlinear partial differential equations for the local thickness. For both density-stratified fluid layers and gravity-driven jets, numerical study of the dynamics at second order suggests strongly the collision of the interfaces in finite time. There are associated singularities both in the fluid velocity and in geometric properties of the interfaces.

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Topology transition is the common and central feature of phenomena as diverse as vortex reconnection in hydrodynamics, the coarsening of soap froths, and the formation of fluid droplets. Figure 1(a) shows a simple physical realization of the latter: the detachment of a pendant drop in a Hele-Shaw cell [1]. This example is especially interesting as the transition involves surfaces material to the flow, and their reconfiguration must entail a singularity in the velocity field. Similar issues have arisen in studies of possible singularity formation in the Euler equation [2,3] and convection in porous media [4].

Here we describe a theory of the approach to topology transition in the Hele-Shaw approximation, wherein viscosity dominates and Darcy's law governs the dynamics. We consider gravity-driven phenomena in two distinct situations involving the interaction of two interfaces, Γ_1 and Γ_2, which bound a thin layer of fluid lying between two other immiscible fluids, as in Fig. 1(b). In the first case, an unstable density stratification, $\rho_1 < \rho_2 < \rho_3$, with gravity pointing normal to the layer, drives the interfaces together. The second, more directly relevant to the experiments, is the gravity-driven jet, with $\rho_1 = \rho_3 < \rho_2$, and gravity pointing along the layer. Our focus is first on questions of geometry such as: Are there singularities in

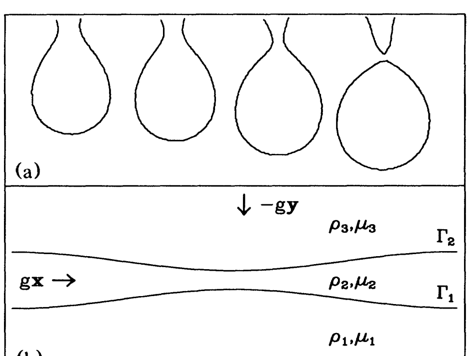

FIG. 1. (a) Detachment of a pendant drop in Hele-Shaw flow, from [1]. (b) Geometry of thin layers in the two cases considered in the text.

the shape of colliding interfaces? Second, we ask about *dynamical* phenomena such as: What is the nature of the fluid velocity field in the neighborhood of the collision?

In addressing these questions, we consider antisymmetric perturbations to the width of the thin layer, and using the aspect ratio of the layer as a small parameter, derive governing partial differential equations (PDE's) for the half-width $h(x,t)$ by a systematic expansion from the boundary integral formalism [5]. These results complement an earlier more heuristic discussion by Constantin *et al.* ([6] referred to as C93), on flows in the absence of gravity. Truncation of these expansions at second order yields an approximation known as *lubrication theory*. In rescaled form, these lubrication equations are

$$h_t(x,t) = -\partial_x\{h(h_{xxx} - Bh_x)\} \quad (1)$$

and

$$h_t(x,t) = -\partial_x\{h(h_{xxx} + B(1+\alpha\mathcal{H}[h_x]))\}\,, \quad (2)$$

for the density-stratified layers and gravity-driven jets, respectively [7]. Here, $\mathcal{H}$ is the Hilbert transform, $\alpha = \mu_1/\mu_2$ is the ratio of fluid viscosities, and the dimensionless "Bond number" B measures the relative importance of buoyancy to surface tension σ,

$$B = 2g\Delta\rho L^2/\sigma\,, \quad (3)$$

where L is a lateral length scale.

The dynamics in (1) and (2) satisfy a continuity equation reflecting the fluid incompressibility

$$h_t + j_x = 0, \quad j = hU\,, \quad (4)$$

where the current j is proportional to the thickness, as in shallow-water theory [8]. The flux form of the equations of motion implies that the vanishing of the layer width h at any point is associated with a divergence of the velocity gradient U_x (see C93). That is, let $X(t)$ locate a minimum of h that reaches zero at a finite time t^*. Then

$$\frac{d}{dt}h(X(t),t) = -h(X(t),t)U_x(X(t),t)\,. \quad (5)$$

It follows then that if there is a finite time t^* such that

$h(X(t),t)\to 0$ as $t\to t^*$, then $U_x(X(t),t)\to +\infty$ as $t\to t^*$. In the presence of surface tension, this implies that at least h_{xxxx} diverges at the interface osculation.

Numerical evidence is presented strongly suggesting that the thickness of the fluid layer can indeed reach zero in finite time (*pinch*), starting from strictly positive initial data. This is interpreted as the signature of a topology transition. At the same time, the simulations reveal finite-time singularities in the shape of the interface.

We begin with the assumption that the fluid velocities $\mathbf{v}_j$ obey Darcy's law,

$$\mathbf{v}_j = -\frac{b^2}{12\mu_j}(\nabla P_j + \rho_j \nabla\phi)\,, \tag{6}$$

where b is the gap width of the cell, P_j, ρ_j, and μ_j are the pressure, density, and viscosity in each fluid, and $\mathbf{F} = -\nabla\phi$ is a divergence-free body force (i.e., gravitational force). By the incompressibility constraint, $\nabla\cdot\mathbf{v}_j=0$, the pressure is harmonic and acts as the velocity potential, and so the velocity field is irrotational. At an interface Γ_j, we impose as a boundary condition that the pressure jump is $P_j - P_{j+1} = -\sigma_j\kappa_j$, where κ_j is the interfacial curvature and σ_j is the interfacial tension. The requirement that the interfaces move with the fluids gives the kinematic boundary condition $\mathbf{n}_j\cdot(\mathbf{v}_j-\mathbf{v}_{j+1})|_{\Gamma_j}=0$, where $\mathbf{n}_j$ is the normal to Γ_j. It follows that the velocity field can be expressed as that arising from a superposition of vortex sheets on Γ_1 and Γ_2 [9]. Adopting a complex representation, $z_j(p,t)=x_j(p,t)+iy_j(p,t)$ is the location of a point on the interface Γ_j, parametrized by a "Lagrangian" variable p. In terms of as yet unknown vortex sheet strengths γ_j, the evolution is given by

$$\frac{\partial z_j^*}{\partial t}(p,t) = \sum_{k=1}^{2}\frac{1}{2\pi i}P\int_{-\infty}^{+\infty}\frac{\gamma_k(p')}{z_j(p)-z_k(p')}dp'\,, \tag{7}$$

for $j=1,2$, where P means principal value (if $j=k$). The γ_j are solutions to the integral equations [5]

$$\gamma_j(p) + 2A_j\,\mathrm{Re}\{\partial_p z_j \partial_t z_j^*\} = \partial_p[S_j\kappa_j(p) - R_j\phi(z_j(p))]\,. \tag{8}$$

Here, lengths have been rescaled by L, the body force $\mathbf{F}$ by g, while $S_j=\sigma_j/\theta_j L^2\sqrt{Lg}$, $R_j=2(\rho_j-\rho_{j+1})g/\theta_j\sqrt{Lg}$, $\theta_j=6(\mu_j+\mu_{j+1})/b^2$, and $A_j=(\mu_j-\mu_{j+1})/(\mu_j+\mu_{j+1})$ is the Atwood ratio of the viscosities.

In applying Eqs. (7) and (8) to the problem at hand, the initial data (and subsequent evolution) are assumed to be *antisymmetric*, as in Fig. 1, for these configurations lead most naturally to the collision of the interfaces. This is equivalent to the problem of a half-layer against a solid wall. The two interfaces are then related by $z_1(p,t) = z_2^*(p,t)$, with $\gamma_2=-\gamma_1\equiv\gamma$. A simple choice of material parameters consistent with this antisymmetry is $\mu_1=\mu_3$, $\sigma_1=\sigma_2$, and $\rho_1-\rho_2=\pm(\rho_2-\rho_3)$, for Eqs. (1) and (2), respectively.

Consider now a layer of length l and area $\mathcal{A}$, whose *mean* thickness is $w=\mathcal{A}/l$. If we write $w=\epsilon l$, then a thin layer means $\epsilon\ll 1$. Moreover, ϵ is a conserved quantity since the area, $\mathcal{A}=\epsilon l^2$, is constant due to the fluid's incompressibility. Thus, we define a rescaled thickness H:

$$h(x,t)=\epsilon H(x,t)\,. \tag{9}$$

Since the vortex sheet strength depends upon the interface curvature, $\kappa=\epsilon H_{xx}/(1+\epsilon^2 H_x^2)^{3/2}$, and its position (9), γ has an expansion in powers of ϵ:

$$\gamma=\gamma^{(0)}+\gamma^{(1)}+\gamma^{(2)}+\cdots\,. \tag{10}$$

It is then straightforward to substitute Eqs. (9) and (10) into (7) and (8) to obtain the interface evolution to any desired order. Care must be taken to account for near singular contributions to the integrals in Eq. (7) [10].

In the Eulerian frame, and assuming that the interfaces $z_2(x,t)=x+ih(x,t)$ and $z_1=z_2^*$ remain single valued in x, the half-width $h(x,t)$ evolves as $h_t(x,t)=v-uh_x(x,t)$, where $u-iv$ is the conjugate velocity on Γ_2. For the cases considered here, we have the central result

$$h_t = -\partial_x\{h(\gamma^{(0)}+\gamma^{(1)}+\gamma^{(2)}+\gamma^{(0)}\mathcal{H}[h_x] + \mathcal{H}[(\gamma^{(1)}h)_x] - \gamma^{(0)}(hh_x)_x\mathcal{O}(\epsilon^3))\}\,. \tag{11}$$

For density-stratified layers, the potential is $\phi(z_2(x)) = h(x)$. The leading terms in the expansion of the vortex sheet strengths are $\gamma^{(0)}=0$ and $\gamma^{(1)}=(Sh_{xxx}-Rh_x)/(1-A_1)$. We then obtain Eq. (1) by absorbing $S/(1-A_1)$ into a rescaled time, with $B=R/S$. The case of the gravity-driven jet, with $\phi(z_2(x))=-x$, is analogous, and by rescaling time as above we obtain Eq. (2).

The sense in which the lubrication theory properly describes interacting interfaces is revealed from a linear stability analysis of a flat interface $h=\bar{h}$. Considering, for example, Eq. (1), the growth rate λ_k of a disturbance of wave vector k is

$$\lambda_k = -\bar{h}(k^4+Bk^2)\,, \tag{12}$$

which is in fact the expansion for small $k\bar{h}$ of the exact result from the full vortex sheet dynamics [5],

$$\lambda_k = -\tfrac{1}{2}(|k|^3+B|k|)(1-e^{-2|k|\bar{h}})\,. \tag{13}$$

In this context, *the lubrication approximation is valid for fluid layers whose modulations are on length scales large compared to their thickness.* The dispersion relation (12) embodies the Rayleigh-Taylor instability of unstably stratified layers ($B<0$), with surface tension stabilizing small wavelengths. The fastest-growing mode has wave vector $k_{\max}=\sqrt{-B/2}$. An analogous stability analysis for the jet shows that the nonlocal contribution from the density stratification gives a dispersive contribution to the growth rate, as found in the Benjamin-Ono equation [8].

In the case of the jet, the advective term Bh_x can be removed by a Galilean transformation if the boundary conditions are open or periodic (as assumed above). But, following previous work (C93) and in light of Fig. 1(a), we also consider a jet which emanates from an aperture with

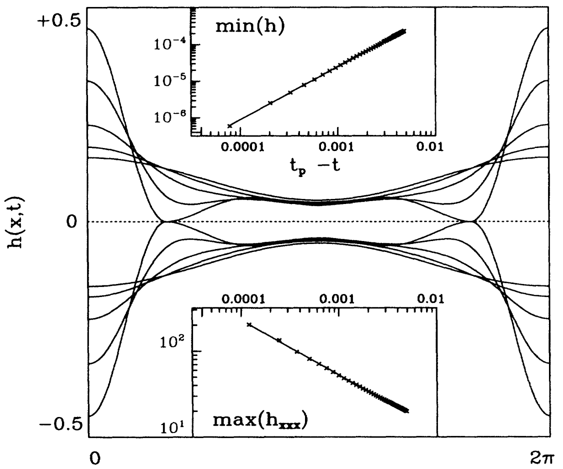

FIG. 2. Interface pinching in density-stratified fluid layers, for $B=-10$. Insets show (top) the vanishing of the minimum thickness, and (bottom) the divergence of the maximum value of the dominant contribution to vortex sheet strength.

fixed width $2h_0$ at $x=0$, forming the neck of a growing drop. With these fixed boundary conditions, the advective term cannot be removed. If the outside fluid is air ($\mu_1=0$), then the nonlocal term in Eq. (2) is not present, and the interface obeys

$$h_t(x,t)=-\partial_x\{h[h_{xxx}+B]\}\,. \tag{14}$$

We may then fix the external pressure P, and impose the Young-Laplace condition $h_{xx}(0,t)=P$. At the bottom of the jet, we have $h(L,t)=h_0$ and $h_{xx}(L,t)=P+BL$, the additional term BL accounting for the increase in pressure due to the greater depth in the fluid. The location $L(t)$ of the bottom boundary is determined by requiring that the increase in area enclosed by the interfaces is from inward flow at the top of the jet, and yields $L_t=h_{xxx}(L,t)+B$.

Numerical solutions of the lubrication equations (1), (2), and (14) elucidate the singularity formation [5]. Implicit pseudospectral methods are used to study Eqs. (1) and (2) under periodic boundary conditions. Figure 2 shows how a sinusoidal perturbation to a thin layer develops through the Rayleigh-Taylor instability to produce pinching. Near the pinching time t_p, the minimum of h (Fig. 2, top inset) fits well to a power law $h_{\min}(t)\sim(t_p-t)^\beta$, with $\beta\simeq1.45\pm0.05$. The accompanying spatial singularity (Fig. 2, bottom inset) fits a divergence of the form $h_{xxx}(t)\sim(t_p-t)^\chi$, with $\chi\simeq0.64\pm0.05$. Further increase in the Bond number yields earlier pinching with structure on smaller scales; the infinite Bond number limit ($S=0$) is actually a very ill-posed evolution for h.

Conversely, for the gravity-driven jet of Eq. (2), the limit of vanishing tension gives a well-posed evolution. Figure 3(a) shows the development of the same initial condition as above in the limit $B\rightarrow\infty$, with $\alpha=1$. In the moving frame used for the illustration, the location of the

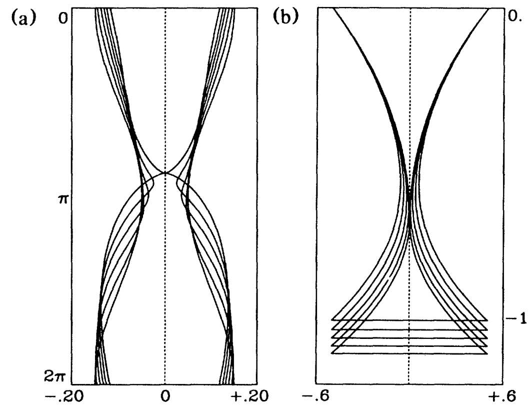

FIG. 3. Dynamics of the gravity-driven jet. (a) The zero surface tension limit with periodic boundary conditions. (b) Local jet dynamics with fixed boundary conditions.

minimum of h actually recedes upwards with time. Here, the pinching is of a much different form; the interface minimum vanishes as a square root in time ($\beta=\frac{1}{2}$), and h_x diverges at the pinch point. In the previous case, the presence of surface tension did not prevent the singularity. Now, however, we find that including surface tension apparently prevents an actual osculation, although the interfaces can come very close. After some interesting transients, the system ultimately relaxes to a flat state.

For the jet dynamics on a finite interval, Eq. (14), an implicit finite-difference scheme yields the dynamics shown in Fig. 3(b), with the moving bottom boundary evident. Once again, we find pinching at a finite time, fitting well to a power-law vanishing of $h_{\min}$ ($\beta\simeq2.0\pm0.1$) and divergence of h_{xxx} ($\chi\simeq1.2\pm0.1$). Note that in each of the three cases the divergence observed is in the velocity U of Eq. (4). This is stronger than necessitated by the argument following Eq. (5), which yields $U_x\rightarrow\infty$ at the pinching time.

In light of the numerical evidence for singularities, we return to the general result (11) to examine the relevance of higher-order terms. For both the unstably stratified layer and local jet dynamics we find that the $\mathcal{O}(\epsilon^2)$ corrections remain small relative to the dominant (h_{xxx}) term even when singularities develop. In addition, for the first case, the behavior shown here is consistent with numerical solutions of the *full* Hele-Shaw equations [5]. On the other hand, in the $B\rightarrow\infty$ limit of the nonlocal jet dynamics, it appears that higher-order terms are important. While the precise effect of these terms is as yet unclear, Eq. (2) will likely prove useful in interpreting the limit of small surface tension, particularly for the outer flow far from the point of pinching.

A careful examination of Eqs. (1) and (2) reveals that the velocities U which enter the continuity equation (4) have the variational form

$$U=\delta\mathcal{C}/\delta h-\partial_x(\delta\mathcal{D}/\delta h)\equiv U_{\mathcal{C}}+U_{\mathcal{D}}\,. \tag{15}$$

For Eq. (1) these functionals are

$$\mathcal{D}=\int dx\{\tfrac{1}{2}Sh_x^2+\tfrac{1}{2}Rh^2\},\quad \mathcal{C}=0\,, \tag{16}$$

while for Eq. (2) they are

$$\mathcal{D}=\int dx\{\tfrac{1}{2}Sh_x^2-Rhx\},\quad \mathcal{C}=\frac{\alpha}{2}\int dx\,Rh\mathcal{H}[h_x]\geq 0\,. \tag{17}$$

The sign of $\mathcal{C}$, for $R<0$, follows from the properties of Hilbert transforms in Fourier space, where $\mathcal{C}=(-\alpha R/2)\sum_{\mathbf{k}}|k||\hat{h}(k)|^2$. When $\alpha=0$ and an external pressure is applied, $\mathcal{D}$ has an additional term $\int dx\,Ph$.

For each of these physical situations, the functional $\mathcal{D}$ is the excess energy associated with the presence of the intervening fluid layer, and its functional derivative with respect to h is the pressure difference acting on the interface. The term $(S/2)h_x^2$ represents the excess surface energy over that of a flat interface, arising from the small-gradient expansion of the metric $(1+h_x^2)^{1/2}$. The terms $(R/2)h^2$ and Rhx in Eqs. (16) and (17) represent the gravitational potential energy of the intermediate fluid layer, associated with columns of fluid of height h or with elements of fluid of area hdx at height x.

If $\mathcal{C}=0$, then $\mathcal{D}$ evolves as

$$\mathcal{D}_t=-\int_{-\infty}^{\infty}dx\,hU_{\mathcal{D}}^2\,. \tag{18}$$

Then, provided the half-width h is positive, $\mathcal{D}$ might act as a Lyapunov functional. However, the energy functionals $\mathcal{D}$ here are not necessarily of definite sign (say, when $R<0$ for the density stratified layers). Still, the numerical studies suggest that in the outer flow, away from the point of nascent singularity, the interface shape is attracted to functional extrema of $\mathcal{D}$, where $U=0$ (see also C93). In the case of Eq. (1), this far-field behavior is associated with the rising spikes that pull fluid out of the central region. Connections between asymptotic behavior near a singularity and extrema of Lyapunov functionals have been seen in the context of semilinear heat equations [11]. This link may prove important in the understanding of the analytic structure of the singularities.

Finally, we remark that the PDE's (1), (2), and (14) may be recast exactly into the scaling form

$$h(x,t)=t^{\psi}f(x/t^{\nu})\,, \tag{19}$$

for particular choices of ψ and ν. The resulting ordinary differential equations are

$$(FF_{zzz})_z-B(FF_z)_z-F=0\,, \tag{20}$$

for the Rayleigh-Taylor problem ($\psi=-1$, $\nu=0$),

$$(FF_{zzz})_z+(B-z)F_z+3F=0\,, \tag{21}$$

for the local jet dynamics ($\psi=3$, $\nu=1$), and likewise for the nonlocal jet equation. While these similarity solutions are exact transformations of the original PDE's, and support singularities, they do not necessarily represent the flow obtained from arbitrary initial conditions. Rather, it appears necessary to construct matched asymptotic expansions which link the large-scale flow far from the pinch point to that in the inner region near the singularity. This problem of energy cascade from large scales to small, and possible associated logarithmic corrections to power-law behaviors, are topics of current interest [12, 13]. Experiments [1] also suggest that inertial effects near the topology transition may influence the detailed shape of the colliding interfaces, and this is an additional area for further work. In the absence of gravity, transitions driven by applied pressure across a thin neck of fluid do show shapes at the pinch point like those predicted by the lubrication theory [6]. Experimental studies of topology transitions resulting from the Rayleigh-Taylor instability considered here should also prove most illuminating.

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