Conformally Invariant Off-Shell String Physics

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Using recent advances in the understanding of noncritical strings, we construct a unique, conformally invariant continuation to off-shell momenta of Polyakov amplitudes in critical string theory. Three-point amplitudes are explicitly calculated. These off-shell amplitudes possess some unusual, apparently stringy, characteristics, which are unlikely to be reproduced in a string field theory. Thus our results may be an indication that some fundamentally new formulation, other than string field theory, will be required to extend our understanding of critical strings beyond the Polyakov path integral.

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Off-shell amplitudes are of great physical interest for string theories, as they are for field theories. They are essential for the derivation of effective actions, e.g., the derivation of effective potentials for particles such as the tachyon and the dilaton, for the derivation of measures for integrating over moduli of space-time instantons in string theory [1], and for the calculation of hadronic form factors when one attempts to interpret certain aspects of quantum chromodynamics in terms of effective string theories. The purpose of the present Letter is to show that there is a unique continuation of Polyakov scattering amplitudes [2] to off-shell momenta, and that recent advances in noncritical string theories [3–5] have rendered computations of these off-shell amplitudes practicable.

In string theory despite intensive investigations in the past, off-shell amplitudes have proven to possess a remarkable intransigence. Space does not permit an extended discussion of previous work here [6], but we provide a summary to put our work in perspective.

(1) The first attempts [7] gave integral formulas for off-shell extensions of the Veneziano amplitude that obeyed various physical criteria such as crossing symmetry, vector dominance of form factors, Regge behavior, and current conservation.

(2) String field theories (SFTs) [8] naturally provide off-shell extensions. Such extensions are not dual, since SFT amplitudes are sums of Feynman diagrams constructed from certain building blocks of fixed geometry, and independence from the conformal frame of these building blocks holds only when all external legs are on-shell. Thus, off-shell SFT amplitudes are not "stringy," and often possess spurious singularities. SFT is an economical way of extending a first-quantized understanding of strings, and the fact that such an extension does not exhibit stringy properties off-shell is no reason for immediate derogation.

(3) Bardakçi and Bardakçi and Halpern [9] considered an off-shell extension while investigating spontaneous symmetry breaking in dual models. They introduced a fictitious dimension, with momenta in this dimension restricted to ± 1. This enabled them to preserve conformal invariance while computing tachyon amplitudes at zero space-time momentum. It will be evident in the following that this work comes closest to the approach based on the Polyakov functional integral [2] (defined below) that we pursue here.

(4) Since Polyakov's work [2], attempts have been made to compute amplitudes on surfaces with boundaries, with off-shell external states specified as matter configurations on the boundaries. Some of this work [10] uses known mathematical results for surfaces with a reflection symmetry, treating surfaces with boundaries in terms of their "double" surfaces, with the boundaries as the curves left invariant under the reflection symmetry. The imposition of physical boundary data, which is naturally independent of the parametrization of the surface, is not treated in these works. Another approach [11] attempted to compute the functional integral directly for a cylinder, but neglected the Weyl dependence in the integration over reparametrization of boundary data. It might be supposed that the pointlike states considered in Ref. [11] should not suffer from this problem [12].

A general feature of approaches (1)–(3) mentioned above is the problem of maintaining off-shell conformal invariance. Recall that Polyakov's derivation of the connection between conformal anomalies and the critical dimensions of string theories [2] elucidated a multitude of features of string physics, gleaned piecemeal in pioneering work. Polyakov's framework proved to provide the foundation for most advances in our understanding of string theories which have followed. It is natural then to suppose that this approach should have something to say about the off-shell properties of string theory. This is indeed the case, as we shall show. Using recent advances in noncritical string theories [3–5], we explicitly calculate an off-shell three-point function. We also compute a tree-level effective action for the tachyon of bosonic string theory.

Within Polyakov's approach, space-time scattering amplitudes of string excitations are calculated as correlation functions of vertex operators in a functional integral over the metric on the string world sheet, and the space-time string configurations [13]:

$$\left\langle \prod_i \int d^2z_i \sqrt{g}\, V_i(z_i) \right\rangle \equiv \int \frac{Dg\, DX}{\mathrm{vol(Diff)vol(Weyl)}} \exp(-S[g,X]) \prod_i \int d^2z_i \sqrt{g}\, V_i(z_i) . \tag{1}$$

The measure is divided by the "volume" of the symmetries of the classical action $S \equiv (8\pi)^{-1} \int d^2z \sqrt{g} g^{ab} \partial_a X^\mu \partial_b X_\mu$, with $\mu = 1, \ldots, D$—namely, diffeomorphisms and local Weyl rescalings on the world sheet. Choosing conformal gauge, $g_{ab} \equiv e^{2\phi} \hat{g}_{ab}(m)$, and fixing diffeomorphisms *à la* Faddeev-Popov, these functional integrals reduce to

$$\int dm \frac{D\phi}{\mathrm{vol(Weyl)}} \frac{DX\, \mathrm{Det}'_{\mathrm{FP}}}{\mathrm{vol(CKV)}} \exp(-S[\hat{g},X]) \prod_i \int d^2z_i \sqrt{\hat{g}(m)}\, V_i(z_i) . \tag{2}$$

Here, CKV stands for the conformal Killing vectors that must be taken into account if the world sheet is a sphere or a torus, and dm denotes the measure for integrating over moduli labeling distinct conformal equivalence classes of metrics on surfaces with one or more handles. Equations (1) and (2) are actually only equivalent if Weyl rescaling survives as a symmetry of the quantum path integral. This requires that $D = 26$ in order to cancel the anomalous dependences on the Weyl field ϕ in the measure factor $DX\,\mathrm{Det}'_{\mathrm{FP}}/\mathrm{vol(CKV)}$ [2]. Also, one must impose various *space-time* mass-shell and polarization/gauge conditions on the external string states to avoid any anomalous Weyl dependences from normal-ordering the vertex operators. Combined these restrictions ensure that ϕ completely decouples from on-shell correlation functions in critical string theory. Then the integration over the Weyl factor cancels against the volume of the group of Weyl rescalings in the denominator [i.e., $\int D\phi/\mathrm{vol(Weyl)} \equiv 1$].

Therefore mass-shell conditions are obtained from requiring Weyl invariance. It follows in the Polyakov approach that the calculation of amplitudes for *off-shell* string states requires the ability to compute correlation functions of vertex operators with an anomalous Weyl dependence in the normalized measure $D\phi/\mathrm{vol(Weyl)}$. One may ask why performing these calculations is such a daunting task. The problem resides in the nonlinearity of the Riemannian metric that defines $D\phi$. The full metric g_{ab} is used to define the norm on infinitesimal changes in the conformal factor

$$(\delta\phi, \delta\phi) = \int d^2x \sqrt{g}\, (\delta\phi)^2 = \int d^2x \hat{g}^{1/2} e^{2\phi} (\delta\phi)^2 ,$$

which then explicitly depends on ϕ. Treating the functional integral over ϕ as a standard quantum field theory requires a translation invariant measure $D_0\phi$, defined by the norm, $(\delta\phi, \delta\phi)_0 = \int d^2x \hat{g}^{1/2} (\delta\phi)^2$. As shown by Mavromatos and Miramontes, and, independently, D'Hoker and Kurzepa [5], these two measures are related in a remarkably simple way,

$$D\phi = D_0\phi \exp\left[S_L - \frac{\mu}{\pi} \int d^2z\, e^{\alpha\phi} \right] , \tag{3}$$

where $S_L \equiv \int d^2z/(6\pi) [\partial\phi \bar{\partial}\phi + \frac{1}{4} \hat{g}^{1/2} \hat{R} \phi]$ and μ is the "cosmological constant." The latter coefficient remains undetermined by their computation, but α is explicitly fixed (see below). This relation was originally conjectured [4] in the study of two-dimensional gravity coupled to conformal matter in conformal gauge. It is important to note that the derivation of Eq. (3) is mathematically entirely independent of the rest of the functional integrals involved. It is valid in noncritical string theory, and equally valid in the context of critical string theory.

The only assumption in the present work will be in treating the correlation functions using the methods of conformal field theory. For noncritical strings, this approach has been verified by comparison with the results of matrix model techniques. The stress tensor deduced from S_L is $T_L = \frac{1}{6} [(\partial\phi)^2 - \partial^2\phi]$, and it is easily checked that the central charge $c_L = 0$. Thus the total central charge (for matter, ghosts and, now, Liouville field) remains zero. The weight of an exponential operator $e^{\beta\phi}$ is $\frac{3}{2}\beta(\beta + \frac{1}{3})$. Off-shell vertex operators V_i of weight (Δ, Δ) are dressed the same way as matter operators in noncritical string theories to produce (1,1) operators $\exp(\beta_\Delta \phi) V_i$ with

$$\beta_\Delta = \frac{1}{6} [\sqrt{25 - 24\Delta} - 1] . \tag{4}$$

This is the unique solution for β_Δ such that $\Delta = 1 \Longrightarrow \beta_\Delta = 0$, which insures that in the on-shell limit, these off-shell amplitudes reduce precisely to the usual on-shell amplitudes. Rather puzzling is the nonanalyticity in this prescription at $\Delta = \frac{25}{24}$, since there is no obvious physical reason to restrict $\Delta \leq \frac{25}{24}$. While one expects cuts in loop amplitudes in field theories, it seems difficult to interpret this nonanalyticity as arising from similar physics. A better understanding is certainly required to extend the applicability of our approach, but for the present, we will restrict our attention to $\Delta \leq \frac{25}{24}$.

The presence of the cosmological constant in Eq. (3) is important for defining the integration over ϕ. Insertions of cosmological constant interaction "cancel" Liouville momentum carried by the off-shell vertex operators, and the background charge term in S_L. However, the treatment of the complete action is rather subtle [14,15]. Here, treating the cosmological constant term as a perturbatively defined interaction, we determine $\alpha = \beta_{\Delta=0}$

$=\frac{2}{3}$. One could consider the other branch of the square root, which gives $\alpha=-1$, but $\alpha=\frac{2}{3}$ may be preferred since then this interaction can be interpreted as a zero-momentum tachyon, hence as obtained from the off-shell continuation of a physical state. Also if used as the area operator of the quantum theory, a vanishing area results in the limit $\phi\to-\infty$, in accord with classical expectations.

Explicit computations can be performed on the two-sphere, using the ideas of Goulian and Li [14] to perform the integral over the constant zero-mode, ϕ_0. The classic calculation of Dotsenko and Fateev [16] can then be used to compute the resulting correlation function, with appropriate analytic continuations along the way [14]. The zero-mode integral is

$$\int d\phi_0 \exp(\tfrac{1}{3}\phi_0 - Ce^{(2/3)\phi_0})\exp(\gamma\phi_0) = \tfrac{3}{2}\Gamma(\tfrac{1}{2}(3\gamma+1))C^{-(1/2)(3\gamma+1)},$$

where $\gamma\equiv\sum\beta_i\equiv\sum\beta(\Delta_i)$, and $C\equiv(\mu/\pi)\int d^2z\exp(\frac{2}{3}\tilde{\phi})$, with $\int d^2z\,\tilde{\phi}=0$. The (not yet normalized) amplitude is now

$$\left\langle\prod_i\int d^2z_i e^{\beta_i\phi}V_i(z_i)\right\rangle = \tfrac{3}{2}\Gamma(-s)\prod_i\int d^2z_i\left\langle C^s\prod_j e^{\beta_j\tilde{\phi}}(z_j)\right\rangle_L\left\langle\prod_k V_k(z_k)\right\rangle_m,$$

where $s\equiv-\frac{1}{2}(3\gamma+1)$, and the subscript $L(m)$ stands for Liouville (matter) expectation values. For three-point functions and positive integer values of s, these correlations were treated by Dotsenko and Fateev [16]. Choosing three tachyon operators, $V_j=\exp(ik_j^\mu X_\mu)$, and fixing their positions $\{z_1,z_2,z_3\}$ at $\{0,\infty,1\}$, yields

$$\mathcal{A}=\tfrac{3}{2}\mu^s\Gamma(-s)\Gamma(s+1)\Delta(\tfrac{1}{3})^s\prod_{i=0}^{3}\prod_{k=0}^{s-1}\Delta(1+2\beta_i+\tfrac{2}{3}k).$$

Here $\Delta(z)\equiv\Gamma(z)/\Gamma(1-z)$, and we have defined $\beta_0\equiv-\frac{1}{6}$, but $\gamma=\sum_{i=1}^3\beta_i$. Using the ideas of Ref. [14], the above formula can be continued to the following expressions:

$$\mathcal{A}=[\mu\Delta(\tfrac{1}{3})]^{-(3\gamma+1)/2}\Gamma\left(\frac{1+3\gamma}{2}\right)\Gamma\left(\frac{1-3\gamma}{2}\right)$$
$$\times(\tfrac{2}{3})^{\gamma-2/3}\prod_{i=0}^{3}\prod_{p=0}^{\gamma-2/3}\Delta(1-3\beta_i+\tfrac{3}{2}p)$$
$$\text{or}\ \times(\tfrac{2}{3})^{5\gamma+4}\prod_{i=0}^{3}\Delta(2\beta_i-\gamma)\prod_{p=0}^{\gamma}\Delta(1-3\beta_i+\tfrac{3}{2}p)$$
$$\text{or}\ \times(\tfrac{2}{3})^{9\gamma+14}\prod_{i=0}^{3}\Delta(2\beta_i-\gamma)\Delta(2\beta_i-\gamma-\tfrac{2}{3})$$
$$\times\prod_{p=0}^{\gamma+2/3}\Delta(1-3\beta_i+\tfrac{3}{2}p), \qquad (5)$$

where γ must be such that the upper limits of the products are integers. Combining all of these formulas, we have results which are valid for $\gamma=n/3$ where n is a positive integer or zero. A more extensive description of the analytic continuations above will appear elsewhere [6]. Analytic continuations similar to those in Eq. (5) were proven to hold for noncritical string theory by Aoki and D'Hoker [17].

It is useful to investigate the analytic structure of these amplitudes when γ is held fixed. Considering the ratio of two such amplitudes (with the same values of γ), one finds that the interesting dependence on β_i resides in, e.g.,

$$\prod_{i=1}^{3}\Delta(2\beta_i-\gamma)\prod_{p=0}^{\gamma}\Delta(1-3\beta_i+\tfrac{3}{2}p).$$

One finds poles and *zeros* depending on the value of β_i individually, and γ as well. Note that the restriction which arose in the discussion of the dressings, $\Delta\leq\frac{25}{24}$, also constrains $\beta_i\geq-\frac{1}{6}$. For a fixed γ, this restricts the number of poles and zeros which actually occur. A case of interest because the particles can all go on-shell is $\gamma=0$, where we find $\prod_{i=1}^3\Delta(1-3\beta_i)\Delta(2\beta_i)$. This expression has poles where $\beta_i\to\frac{1}{3}$ (i.e., $k_i^2\to\frac{4}{3}$), and no zeros—in particular, it remains finite as $\beta_i\to0$. A striking feature of the amplitudes is the presence of poles that are not accounted for by excitations in the matter sector (even if combined with the ghost sector). They may indicate the presence of excitations that are entirely stringy in nature.

Independent of the existence of new poles, the fact that the amplitudes have products which have upper limits determined by γ is something entirely unlike the amplitudes one obtains from a field theory. In field theories, the off-shell character of the amplitude is a function of individual external states. Here, one can obtain the value $\gamma=0$ when all external states are on-shell, *or* if they are off-shell. It is difficult to imagine how this γ dependence could be reproduced in a string field theory. Thus our results may be an indication that some fundamentally new framework, other than string field theory, will be required to extend our understanding of critical string theory beyond the Polyakov path integral. Of course, even though our present knowledge of string theory is derived almost entirely from the Polyakov functional integral, it is not possible to exclude the possibility that the Polyakov approach is just a recipe for on-shell calculations.

An important feature which distinguishes our amplitudes from those of noncritical string is the factor vol(Weyl) in the denominator of Eq. (2). The computation of the Weyl volume is subtle. Reference [14] gives a prescription which uses Eq. (5) with $\gamma=2$ and $\beta_i=\frac{2}{3}$ to give a result for the two-sphere (which actually vanishes). At tree level it is possible to evade a direct computation of the Weyl volume by considering ratios of amplitudes. On higher genus surfaces, the presence of this factor ensures that the Weyl field does not show up in any counting of states via degenerations. In particular, the dependence on the moduli in $D\phi$ is precisely canceled by the denominator, unless there are off-shell vertex operators present. Note then that in Eq. (2), dm and $D\phi/\mathrm{vol(Weyl)}$ must be explicitly ordered as given.

In conclusion, we have shown in this Letter that the effort expended on the study of noncritical strings in somewhat unphysical contexts has important physical consequences in critical string theories. It follows as well that any new future insights into noncritical string physics, or into quantum Liouville theory, will translate directly into further insights into off-shell critical string physics. There are a great many physical questions that become accessible in our approach to off-shell string physics. Above we have only considered simple exponential dressings, but one can also find many new (1,1) primary fields with oscillator contributions (e.g., $\partial\phi$) which will couple in amplitudes. Some of these may account for longitudinal polarizations which only couple off-shell. It is possible to derive explicit formulas for four-point functions with mild kinematic restrictions, and of course, a supersymmetric extension of these ideas is immediate. Extending Eq. (5) to arbitrary values of γ is required, as is an understanding of the nonanalyticity in Eq. (4). The calculation of the effective action requires care. Computations for zero-momentum tachyons do not contain expected dilation exchange singularities, and yield a *nonanalytic* tree-level effective potential,

$$\Gamma(T) \sim 3T^{1/3} - T .$$

An extended treatment of these issues is in preparation [6].

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