Black Strings and p-Branes are Unstable

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We investigate the evolution of small perturbations around black strings and branes which are low energy solutions of string theory. For simplicity we focus attention on the zero charge case and show that there are unstable modes for a range of time frequency and wavelength in the extra $10-D$ dimensions. These perturbations can be stabilized if the extra dimensions are compactified to a scale smaller than the minimum wavelength for which instability occurs and thus will not affect large astrophysical black holes in four dimensions. We comment on the implications of this result for the cosmic censorship hypothesis.

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Black holes are perhaps the most puzzling objects in general relativity. They hide behind their horizon a singularity: a point which implies the demise of the theory itself. The area surrounding this singularity is a region of extremely strong gravity, and presumably is described by quantum gravity. In four spacetime dimensions black holes are stable: Once formed they settle down to a state described solely by their mass, charge, and angular momentum; therefore the singularities remain hidden from distant observers. This classical stability of black holes led to Penrose's cosmic censorship hypothesis [1] claiming that all singularities are hidden. Quantum mechanically, black holes are quite different objects; they are analogous to a thermal system. The surface area of the hole behaves like entropy, and it is even possible to associate a temperature to a black hole, as Hawking has shown they radiate thermally [2]. However, Hawking conjectured [3] that a black hole formed from a pure quantum state would radiate away, leaving a mixed state of radiation; this would violate quantum mechanical unitarity. It is difficult to understand the final stage of black hole evaporation as general relativity is expected to break down at Planckian curvatures, but if quantum gravity preserves unitarity and information is to be returned, it must do so well before the black hole reaches Planckian curvatures, otherwise there is simply not enough energy left in a Planck-mass black hole to emit all the information stored in a macroscopic black hole.

Recently, there has been a resurgence of interest in this problem, largely due to the rise of string theory as a candidate for this unified quantum theory. Many efforts have concentrated on the weak gravity regime, analyzing the implications of low energy string theory on black hole structure. Already some of these discoveries have been exciting. In Einstein gravity, charged black holes (the Reissner-Nordström solutions) have an unfortunate weakness. As well as an outer event horizon, they contain an inner Cauchy horizon which is unstable to matter perturbations in the exterior spacetime [4]. However, there is no static charged black hole solution in Einstein gravity with only one horizon and a spacelike singularity. On the other hand, in low energy string theory, gravity acquires a dilaton which greatly changes the causal structure of charged black holes making them Schwarzschild type with one event horizon and a spacelike singularity [5]. This structure is generic, even if the dilaton has a mass [6], as it must do to keep in line with the principle of equivalence. A particularly amusing aspect of these black holes is that in the extremal limit of a magnetically charged black hole, the spacetime acquires an internal null infinity at "$r=2M$" which is an infinite volume "throat" in which much information can be stored.

Of course, all of these models live in low dimensions, whereas string theory tells us there are ten dimensions. Ideally therefore, one should be examining black holes in ten dimensions. There has been work on black holes in higher dimensions [7], including work that allows for a range of horizon topologies [8]. In four dimensions, an event horizon must be topologically spherical [9], but in higher dimensions this is not necessarily the case; we could have $S^2\times\mathbb{R}^6$ or $S^3\times\mathbb{R}^5$ topologies for the horizon. The purpose of this Letter is to point out that a large class of these black holes are unstable under small perturbations. This is a property which is very different from their analog in four dimensions. However, there is a heuristic argument to show that this is reasonable. Consider a five-dimensional black string, Sch$\times\mathbb{R}$. A portion of length L has mass $\mathcal{M}=ML$ and entropy proportional to $\mathcal{M}^2/L$. A five-dimensional black hole, on the other hand, has entropy proportional to $\mathcal{M}^{3/2}$. Thus for large lengths of horizon, the mass contained within the horizon contributes a much lower entropy than if it were in a hyperspherical black hole. This indicates that for large wavelength perturbations in the fifth dimension, we might expect an instability.

The issue of the stability of the five-dimensional black string has been investigated analytically [10] with the result that there is no nonsingular single unstable mode on a Schwarzschild time $t=0$ surface; however, this argument did not prove stability. As emphasized by Vishveshwara [11] in his original Schwarzschild stability argument, the nonexistence of a single unstable mode does not preclude the existence of a composite unstable mode, with the combination canceling the singular behavior of an inadmissable single singular mode. This is in fact the situation with the colored black hole instability, recently confirmed by Bizon and Wald [12]. That this is indeed the situation for black strings was first indicated by Whitt [13], who analyzed four-dimensional fourth order gravity and found an instability—a different physical situation, but mathematically identical equations to those studied in Ref. [10]. The key simplification Whitt found useful was to use a different initial data surface ending on the future horizon. By avoiding the neck of the Schwarzschild wormhole, one avoids the fixed point of the isometries used to generate the mode decomposition, which avoids in this case issues of superposition. By adapting and generalizing his approach, we have been able to show that extended uncharged black p-branes are unstable. It is worth stressing that this instability is not of the Reissner-Nordström form—hidden behind the event horizon—but it is a real *physical* instability of the exterior spacetime which could potentially fragment the horizon. It is important to emphasize that this can occur classically, for although under regular conditions horizons do not bifurcate [14], if one has a naked singularity, then bifurcation is possible. Since an instability calculation is by its nature linear, it cannot predict the end point of an unstable evolution. However, the entropy argument does lend support to the fragmentation scenario and violation of the cosmic censorship hypothesis.

In order to prove linear instability, an analysis of the perturbation equations is required, with suitable reference to gauge and boundary conditions. Although this process is quite detailed and involved, it is nonetheless possible to present the salient features of the argument briefly. This is what we will now do.

The black branes we are specifically interested in are those introduced by Horowitz and Strominger in ten-dimensional low energy string theory with a metric of the form

$$ds^2=-V\,dt^2+\frac{1}{V}dr^2+r^2d\Omega^2_{D-2}+dx^i dx_i\,, \qquad (1)$$

where $V=1-(r_+/r)^{D-3}$, $D=4,\ldots,10$ and the index i runs from 1 to $10-D$. As we are only addressing uncharged black holes here, it is sufficient to consider perturbations to the Einstein equations, since the dilaton and gauge perturbations decouple and can be set to zero. In the usual fashion we write a perturbation of the metric as

$$g_{ab}\to g_{ab}+h_{ab}\,, \qquad (2)$$

where we use the transverse trace-free (de Donder) gauge for h_{ab}:

$$h^a_a=0=h^a_{b;a}\,. \qquad (3)$$

This does not eliminate all of the gauge freedom, but does simplify the perturbation equations

$$\Delta_L h_{ab}=(\delta^c_a\delta^d_b\Box+2R_a{}^c{}_b{}^d)h_{cd}=0\,, \qquad (4)$$

where Δ_L stands for the Lichnerowicz operator.

In general relativity, physics is invariant under general coordinate transformations (GCT's), which are generated by vector fields ξ^a. The effect of an infinitesimal GCT is to push the coordinates ϵ along the integral curves of ξ^a. Under such a gauge transformation, the metric transforms as

$$g_{ab}\to g_{ab}+2\xi_{(a;b)}\,; \qquad (5)$$

hence a pure gauge perturbation of the metric is of the form

$$h_{\xi ab}=2\xi_{(a;b)}\,. \qquad (6)$$

But if ξ^a is divergence free and harmonic, then h_ξ satisfies both (3) and (4). Therefore, although there are $(N-2)(N+1)/2$ degrees of freedom in the solutions to the N-dimensional Lichnerowicz equation, $N-1$ of these are pure gauge, the remaining $N(N-3)/2$ being physical. It will turn out to be fairly straightforward to identify the gauge degrees of freedom.

Now, most importantly, there is the question of boundary conditions, which are the key to this problem. Obviously, we want to place initial data on a Cauchy surface for the exterior spacetime, but such a surface necessarily touches the horizon, which is singular in Schwarzschild coordinates. There are therefore two issues here: One is how to define "small" for the perturbation at the horizon, and second, which initial data surface to impose these constraints upon.

The first issue is straightforwardly dealt with. Although the horizon is singular in Schwarzschild coordinates, it is not a physical singularity, merely a coordinate singularity. In four dimensions, nonsingular coordinates have been known for some time—Kruskal coordinates. These require generalizing to higher dimensions, which is slightly more involved, but the transformation laws between Kruskal and Schwarzschild coordinates remain qualitatively the same at the horizon. Therefore, since Kruskal coordinates do not display their staticity in a straightforward manner, we perform a mode decomposition in Schwarzschild coordinates, transforming to Kruskal coordinates at the horizon to decide which modes are well behaved.

This leaves us with the problem of an initial data surface. The domain of dependence must obviously include $\mathcal{I}^+$; thus a surface touching the future horizon, or the neck of the Schwarzschild wormhole, is acceptable, but a surface touching the past horizon is not, unless it passes through and extends to the opposite horizon on the Penrose diagram. We choose the data surface ending on the future horizon for two reasons. One is that it avoids the issue of mode superposition discussed earlier, and second, we believe it to be a better physically motivated choice of surface. This is because in practice a black hole (or brane) would form in a collapse situation, and hence would not have a Schwarzschild wormhole; analyzing the stability would necessarily require a surface ending on a future event horizon.

Now we turn to the actual stability analysis: Are there any unstable modes? As a result of the symmetries of the spacetime, we can split up the perturbation into a purely transverse piece, a mixed transverse and D-Schwarzschild piece, and a purely Schwarzschild piece. This can be represented schematically as

$$\begin{bmatrix} h_{\mu\nu} & h_{\mu i} \\ h_{j\nu} & h_{ij} \end{bmatrix}, \tag{7}$$

where μ runs from 1 to D. In a Kaluza-Klein spirit, we can interpret these perturbations as scalar, vector, and tensor, respectively, with respect to the D-dimensional Schwarzschild spacetime.

It is relatively straightforward to show that there are no unstable modes with nonzero scalar or vector pieces meeting our criteria of being well behaved at both infinity and the future event horizon. However, for a D-dimensional s wave of the form

$$h^{\mu i} = 0 = h^{ij},$$

$$h^{\mu\nu} = e^{\Omega t} e^{i\mu_i x^i} \begin{bmatrix} H^{tt} & H^{tr} & 0 & 0 & \cdots \\ H^{tr} & H^{rr} & 0 & 0 & \cdots \\ 0 & 0 & K & 0 & \cdots \\ 0 & 0 & 0 & \dfrac{K}{\sin^2\theta} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix} \tag{8}$$

for certain values of Ω and $\Sigma\mu_i^2$, a solution to the Lichnerowicz equation exists.

Using the metric (1) and the perturbation in the form (8) the Lichnerowicz equation reduces to

$$\left(\Delta_L^D + \sum_i \mu_i^2\right) h_{\mu\nu} = 0\,, \tag{9}$$

where Δ_L^D is the D-dimensional Lichnerowicz operator. Note that a pure D-dimensional gauge perturbation, $h_{\mu\nu} = \xi_{(\mu;\nu)}$, satisfies $\Delta_L^D \xi_{\mu;\nu} = 0$. Thus a pure gauge perturbation of the metric must be a zero mode of the D-dimensional Lichnerowicz equation. Therefore as long as $\mu^2 = \sum_i \mu_i^2 \neq 0$ in Eq. (9), $h_{\mu\nu}$ will be a real physical perturbation.

To find the equation obeyed by the perturbation we use the gauge conditions to eliminate all but one variable from the Lichnerowicz equation, H^{tr}, say, leaving a second order ordinary differential equation:

$$\begin{aligned}
0 = {} & \left\{ -\Omega^2 - \mu^2 V + \frac{(D-3)^2 (r_+/r)^{2(D-3)}}{4r^2} \right\} H^{tr\prime\prime} \\
& - \left\{ \mu^2 \left[(D-2) - 2\left(\frac{r_+}{r}\right)^{D-3} + (4-D)\left(\frac{r_+}{r}\right)^{2(D-3)} \right] + \frac{\Omega^2 [(D-2) + (2D-7)(r_+/r)^{D-3}]}{rV} \right. \\
& \left. + \frac{3(D-3)^2 (r_+/r)^{2(D-3)} [(D-2) - (r_+/r)^{D-3}]}{4r^3 V} \right\} H^{tr\prime} \\
& + \left\{ \left(\mu^2 + \frac{\Omega^2}{V} \right)^2 + \frac{\Omega^2 [4(D-2) - 8(D-2)(r_+/r)^{D-3} - (53 - 34D + 5D^2)(r_+/r)^{2(D-3)}]}{4r^2 V^2} \right. \\
& + \frac{\mu^2 [4(D-2) - 4(3D-7)(r_+/r)^{D-3} + (D^2 + 2D - 11)(r_+/r)^{2(D-3)}]}{4r^2 V} \\
& \left. + \frac{(D-3)^2 (r_+/r)^{2(D-3)} [(D-2)(2D-5) - (D-1)(D-2)(r_+/r)^{(D-3)} + (r_+/r)^{2(D-3)}]}{4r^4 V^2} \right\} H^{tr}\,.
\end{aligned} \tag{10}$$

By inspection of this equation, the regular solution at infinity is $e^{-(\Omega^2+\mu_i^2)^{1/2} r}$, and the solutions at the horizon behave as $(r-r_+)^{-1 \pm r_+\Omega/(D-3)}$. Our boundary conditions demand the positive root and $\Omega > 0$. For $\Omega > (D-3)/r_+$ and any value of μ, we can rule out the existence of instabilities analytically. Unfortunately it is exactly when Ω is of the order of $1/r_+$ that we expect a possible instability. For small Ω and μ_i we can confirm the existence of regular unstable solutions numerically. Obviously because the horizon is singular this process is delicate; however, if we integrate in a regular solution from infinity, the general solution near the horizon will be

$$A_+(\mu)(r-r_+)^{-1+r_+\Omega/(D-3)} + A_-(\mu)(r-r_+)^{-1-r_+\Omega/(D-3)}\,. \tag{11}$$

By taking appropriate combinations of this function and its derivative, we determine the ratio $R = A_-/A_+$. Existence of a solution (and hence an instability) is determined by a zero of R. We observe this in practice by a change in sign of R, for which R decreases as we home in on the sign change. An increase in R would indicate a zero of A_+. We found that there did indeed exist zeros of R for a range of Ω, for all $4 \leq D \leq 9$, with appropriate

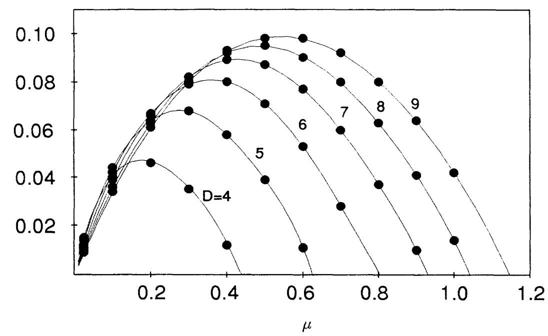

FIG. 1. Plot of Ω as a function of μ for black strings and branes with $D=4,\ldots,9$ for which an instability has been found. The bold points correspond to values calculated numerically and the lines have been traced to guide the eye.

values of μ shown in Fig. 1. The points in Fig. 1 correspond to the values calculated numerically and the lines have been added to guide the eye. There is a symmetry in the equation for H^{tr} under the following transformation: $r_+ \rightarrow \alpha r_+$, $\Omega \rightarrow \Omega/\alpha$, and $\mu \rightarrow \mu/\alpha$, for a constant value of α. Thus it is sufficient to calculate Ω and μ for only one value of r_+.

The significance of these results is easily summarized: Black strings and branes are classically unstable. This is a real instability, for clearly the perturbation cannot be written as pure gauge. By exhibiting a *single* (Ω,μ) for any black brane we prove instability, and by exhibiting a range we indicate the instability is generic and robust. How might we interpret this result physically? Of course, since our calculation is linear, we cannot strictly say anything about the final state, but the entropy argument, as well as the fact that h_{ab} dominates g_{ab} in Schwarzschild coordinates near the horizon, makes it tempting to suggest that the black brane will fragment. Periodic black hole solutions are known [15], so there is a known final state solution in this case (unlike the Reissner-Nordström case). Such a process will produce a naked singularity and hence violate cosmic censorship. Perhaps a more realistic though less spectacular conclusion is that due to this instability, black strings and p-branes will not form in the first place from collapse.

The only way around the instability is to compactify the transverse dimensions on a scale smaller than the inverse mass of the black hole. The compactification would imply that the values of μ_i are quantized. If their first value is greater than the maximum one in Fig. 2 this would imply that such "black doughnuts" would be stable. Since there must be compactification of any extra dimensions on an extremely small scale, all but the tiniest black doughnuts would be safe, and those that would not would presumably have evaporated producing their own naked singularities long ago. Thus this instability will have no effect for contemporary astrophysical black holes.

Naturally this work makes no statement about classically charged black holes. An investigation into these is in progress. Although the true end point of this instability is not presently known, it would have important consequences for the cosmic censorship hypothesis. The form of δg_{ab} indicates that these perturbations add an oscillatory component to the location of the horizon as a function of x_i (the extra dimensions). If these instabilities lead to a shrinking of the event horizon, black hole singularities might reveal themselves. A generic regular initial perturbation would therefore develop into a visible singularity. The extremal case, where the event horizon and singularity coincide, is of particular interest. If the event horizon shrinks, even by a very small amount, this instability may lead directly to a naked singularity. This case is under present investigation.

Finally, to reiterate our original theme, this result makes clear the domain of validity of four-dimensional Einstein gravity—namely, four-dimensional Einstein gravity. The stability of four-dimensional Schwarzschild black holes does not imply that five-dimensional black strings or ten-dimensional black branes are stable—indeed they are not. The result highlights the unexpected subtleties of black holes, and is a demonstration that an event horizon too can be ephemeral.

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