Probability Densities of Turbulent Temperature Fluctuations

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(Received 24 August 1992)

Starting from an assumption of universality in turbulence and building upon the work in Y. G. Sinai and V. Yakhot, Phys. Rev. Lett. **63**, 1962 (1989), we construct a closed-form expression for the probability density function of temperature fluctuations. This result is found to be in good correspondence to experimental data obtained at Chicago and Yale. By extending this method, we obtain a similar expression for the probability density function of temperature differences between two times. Again the result is checked to hold very well, except for very short time separations.

PACS numbers: 47.27.−i, 02.50.−r, 05.40.+j

Probability density function (PDF) of fluctuations is of increasing interest in turbulence. With a view to studying possible corrections to Kolmogorov's five-thirds law and the related intermittency problem, a lot of experimental [1] and numerical [2] data for derivatives and differences of velocity and scalar fields have been accumulated. The PDF's of these quantities are found to be non-Gaussian. Recent Chicago experiments [3] on Rayleigh-Bénard convection in low-temperature helium gas revealed that there are two qualitatively different turbulent states. In the soft-turbulence regime [Rayleigh number (Ra) $\lesssim 10^8$], the PDF of temperature fluctuations measured at the center of the experimental cell is Gaussian, while in the hard-turbulence regime (Ra $> 10^8$), the PDF is non-Gaussian and close to exponential (sometimes high-pass filtering [4] is required to get the exponential). The PDF's of temperature differences, between two different times, were also studied [5] and found to be quite well approximated by a stretched exponential: $e^{-c|x|^\beta}$. The values of β become smaller towards shorter time separations.

The discovery of nearly exponential temperature fluctuations in convection has stimulated some recent theoretical [6–10] and experimental [11,12] studies on PDF's. The diffusion of a temperature field, $T(\mathbf{x},t)$, which simultaneously undergoes advection in an incompressible velocity field, $\mathbf{u}(\mathbf{x},t)$, is described by the following equation:

$$\frac{\partial T}{\partial t}+\mathbf{u}\cdot\nabla T=\kappa\nabla^2 T\,, \qquad \nabla\cdot\mathbf{u}=0\,, \tag{1}$$

with κ being the thermal diffusivity. Equation (1) is valid when temperature acts as a passive scalar as well as in thermal convection in which temperature difference drives the flow. In the latter case, there is another equation, the Boussinesq equation [13], which couples the velocity and the temperature fields. From Eq. (1), Sinai and Yakhot [6] derived a "fluctuation-dissipation" relation for the case when the system is homogeneous and the temperature field is not forced:

$$\langle X^{2n}\rangle=(2n-1)\langle X^{2n-2}\tilde{Y}^2\rangle\,. \tag{2}$$

Here $X=(T-\langle T\rangle)/\langle(T-\langle T\rangle)^2\rangle^{1/2}$ is the temperature fluctuation normalized by its own standard deviation and $\tilde{Y}=\nabla T/\langle(\nabla T)^2\rangle^{1/2}$ is the normalized temperature gradient. Their analyses are thus not directly applicable in the case of thermal convection when the system is forced (a nonzero temperature difference is maintained) nor even when temperature is passive as external heat is supplied to maintain the fluctuations for measurements. Using Eq. (2), Sinai and Yakhot obtained a closed form for the asymptotic PDF of the decaying temperature fluctuations in homogeneous systems.

In this paper, we assume that a fluctuation-dissipation relation similar to Eq. (2) holds in general for turbulent temperature fluctuations, namely,

$$\langle X^{2n}\rangle=(2n-1)\langle X^{2n-2}Y^2\rangle\,, \tag{3}$$

where $Y=(\partial T/\partial t)/\langle(\partial T/\partial t)^2\rangle^{1/2}$ is the normalized temperature derivative. (The variables Y and $\tilde{Y}$ have similar statistics when a mean flow exists and Taylor's frozen flow hypothesis [14] is valid [15].) The use of Eq. (3) in situations where the system is forced is not obvious and might be justified by considerations of universality of fluctuations in turbulence. Then following Sinai and Yakhot [6], we obtain a closed-form expression for the PDF of X, denoted as $P(x)$. The expression depends only on one quantity, $q(x)$, which is defined by

$$q(x)\equiv\frac{\langle(\partial T/\partial t)^2\rangle_{X=x}}{\langle(\partial T/\partial t)^2\rangle}\,. \tag{4}$$

The subscripts $X=x$ indicate that the mean square is to be calculated at a given value, x, of the normalized temperature fluctuation X. Our result is

$$P(x)=\frac{C}{q(x)}\exp\left[-\int_0^x\frac{x'dx'}{q(x')}\right]. \tag{5}$$

The constant C is not arbitrary but fixed by the definition of PDF: $\int P(x)dx=1$. Note that Eq. (5) explicitly relates the PDF of temperature fluctuations to the statistics of temperature derivative.

Equation (5) can be tested with the Chicago convective turbulence data [3]. We compute the PDF and q directly from the temperature data taken at the center of the experimental cell. Then we compare the PDF directly from the data, which we denote as P_{exp}, with the PDF obtained from q using Eq. (5), which we denote as P_{th}. Figure 1 shows one such comparison for $Ra = 5.8\times 10^{14}$, which is in the hard-turbulence regime. We see that P_{exp} and P_{th} agree very well. The bigger fluctuation of $P_{th}(x)$ for large $|x|$ reflects the larger statistical error of $q(x)$ there (due to fewer observations of large $|x|$). To measure how good the agreement is, we define a quantity η by

$$\eta \equiv \left[\frac{\int \Delta^2(x) P_{exp}^{1/2}\,dx}{\int P_{exp}^{1/2}\,dx} \right]^{1/2}, \tag{6a}$$

where

$$\Delta(x) = \log_2 \left[\frac{P_{th}(x)}{P_{exp}(x)} \right]. \tag{6b}$$

The weight factor $P_{exp}^{1/2}$ is included in the definition of η to take into account the statistical error in computing P_{th}. A plot of $\Delta(x)$ is shown in the inset of Fig. 1. In Table I, we display the values of η for nine Ra's which range from 10^7 to 10^{15}. It can be seen that good agreement is found, in general, for both soft and hard turbulence, with P_{th} and P_{exp} always within a factor of 2 of each other ($\eta < 1$). Hence, Eq. (5) works well for the convective turbulence data. We emphasize that this result is not a simple generalization of Sinai and Yakhot's work [6], since their result is derived only in the case of decaying temperature fluctuations. On the other hand, Yakhot [7] obtained an expression different from Eq. (5) for PDF's

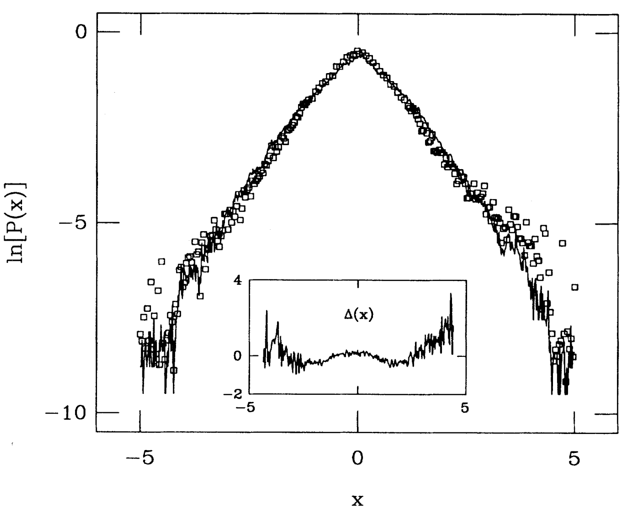

FIG. 1. Testing the validity of Eq. (5) for the PDF of the normalized temperature fluctuations, $P(x)$, using convective turbulence data with $Ra = 5.8\times 10^{14}$. The solid line is the natural logarithm of P_{exp}, the PDF computed directly from the data. The squares are P_{th}, the PDF obtained from $q(x)$ using Eq. (5). It can be seen that they agree well with each other. The inset is $\Delta(x)$ [see Eq. (6b)], which is the logarithm (base 2) of the ratio of the two PDF's.

TABLE I. Values of η [see Eq. (6)] for PDF's of temperature fluctuations and temperature differences. τ is measured in experimental sampling intervals. For those rows without a value for τ, the data are for the PDF of temperature fluctuations. The smaller the value of η, the better Eqs. (5) and (11) can describe the experimental data.

Nature of data	Ra/Re	τ	η
	6.9×10^6		0.26
	2.1×10^7		0.30
	6.0×10^8		0.44
	4.0×10^9		0.48
	7.3×10^{10}		0.45
	6.0×10^{11}		0.43
Convective	6.7×10^{12}		0.39
	4.1×10^{13}		0.36
(Fig. 1)	5.8×10^{14}		0.33
	6.0×10^8	1	0.82
[Fig. 2(a)]	7.3×10^{10}	1	0.79
	7.3×10^{10}	8	0.35
[Fig. 2(b)]	7.3×10^{10}	64	0.30
	7.3×10^{10}	512	0.27
	5.8×10^{14}	1	0.52
	5.8×10^{14}	8	0.26
	5.8×10^{14}	64	0.26
	5.8×10^{14}	512	0.18
	9.5×10^4		0.34
Passive	9.5×10^4	1	0.58
	9.5×10^4	64	0.24

of stationary temperature fluctuations. However, his derivation assumed a constant, nonzero gradient in the mean temperature profile and is thus irrelevant for the case of convection.

Using Eq. (5), we see that a difference between soft and hard turbulence can be attributed to the different behaviors of $q(x)$. Note that $P(x)$ is Gaussian if $q(x)$ is independent of x [$q(x)$ is then identical to 1, since $\int P(x)q(x)dx = 1$ by definition]. On the other hand, if $q(x) \sim x$ then $P(x)$ is an exponential distribution. A possible way of getting $q(x)$ increasing with x is to have a spiky temperature time series. The spikes may be caused by thermal plumes and other coherent structures observed in flow visualizations [16]. This observation agrees with the experimental demonstration by Solomon and Gollub [17] that the form of the PDF's depends heavily on the coherence of plumes.

We then test Eq. (5) by using some temperature data measured in the wake of a heated cylinder [18]. Water of speed 5 m/s was flowed past a heated cylinder of diameter 19 mm (Reynolds number, $Re = 9.5\times 10^4$). Temperature was measured at a fixed point downstream of the cylinder on the wake centerline. The cylinder was heated so slightly that the buoyancy term was unimportant and temperature acted as a passive scalar. Good agreement between P_{th} and P_{exp} is again found. The value of η for this case is also given in Table I.

Next, we extend the method of Sinai and Yakhot to study the PDF's of temperature differences, $T_\tau(x,t) \equiv T(x,t+\tau)-T(x,t)$, between two different times separated by a time interval τ. The equation of motion for T_τ is

$$\frac{\partial T_\tau}{\partial t}+\bar{\mathbf{u}}\cdot\nabla T_\tau+\mathbf{u}_\tau\cdot\nabla\bar{T}=\kappa\nabla^2 T_\tau\,,\qquad \nabla\cdot\mathbf{u}=0\,. \tag{7}$$

Here, the subscript τ denotes the difference of the quantity between times $t+\tau$ and t while the overbar denotes averages of the quantity between the two times. Multiplying Eq. (7) by $2nT_\tau^{2n-1}$ and taking spatial averages, we have

$$\frac{\partial\langle T_\tau^{2n}\rangle}{\partial t}+2n\langle T_\tau^{2n-1}\mathbf{u}_\tau\cdot\nabla\bar{T}\rangle = -2n(2n-1)\kappa\langle T_\tau^{2n-2}(\nabla T_\tau)^2\rangle\,. \tag{8}$$

Note that Eq. (8) is also valid for an incompressible velocity field with a nonphysical impenetrable, free slip boundary condition, used in some numerical simulations [19].

Suppose we can neglect the term $\langle T_\tau^{2n-1}\mathbf{u}_\tau\cdot\nabla\bar{T}\rangle$ (which is identically zero if the joint PDF of T_τ and $\mathbf{u}_\tau\cdot\nabla\bar{T}$ is symmetric in T_τ), then by asserting that the asymptotic PDF's for the normalized temperature differences, $X_\tau\equiv T_\tau/\langle T_\tau^2\rangle^{1/2}$, exist, we obtain the following relation:

$$\langle X_\tau^{2n}\rangle=(2n-1)\langle X_\tau^{2n-2}\tilde{Y}_\tau^2\rangle\,, \tag{9}$$

where $\tilde{Y}_\tau=\nabla T_\tau/\langle(\nabla T_\tau)^2\rangle^{1/2}$. Equation (9) is valid when fluctuations are decaying $(d\langle T_\tau^{2n}\rangle/dt\neq 0)$ and the system is homogeneous (so that PDF can be computed by spatial averaging). Again, we assume that a similar relation holds in general; that is, we assert that

$$\langle X_\tau^{2n}\rangle=(2n-1)\langle X_\tau^{2n-2}Y_\tau^2\rangle\,, \tag{10}$$

where $Y_\tau=(\partial T_\tau/\partial t)/\langle(\partial T_\tau/\partial t)^2\rangle^{1/2}$. Then we get an expression for the PDF's of temperature differences T_τ,

$$P_\tau(x)=\frac{C_\tau}{q_\tau(x)}\exp\left[-\int_0^x\frac{x'dx'}{q_\tau(x')}\right], \tag{11}$$

where q_τ is defined as follows:

$$q_\tau(x)\equiv\frac{\langle(\partial T_\tau/\partial t)^2\rangle_{X_\tau=x}}{\langle(\partial T_\tau/\partial t)^2\rangle}\,. \tag{12}$$

As before, the constants C_τ's are set by the normalization condition. The subscripts $X_\tau=x$ indicate that the mean square is to be calculated at a given value x of X_τ, the normalized temperature difference.

We compute P_τ and q_τ from the data and test Eq. (11) as we did with Eq. (5). It is found that Eq. (11) holds very well for both the convective and passive temperature data, except for very short time separations. Comparisons similar to Fig. 1 are shown in Figs. 2(a) and 2(b) for $Ra=7.3\times10^{10}$ with $\tau=1$ and 64 experimental sampling intervals, respectively. Noticeable discrepancy is observed in Fig. 2(a). However, if one forgoes the correct normalization, the squares can be made (by a translation of the solid line in the log-linear graph) to lie on top of the solid line for large $|x|$. Thus, even in the case of very short time separations, the form of Eq. (11) still works well for large fluctuations. The agreement is better for longer time separations, as can be seen from the smaller values of η [Eq. (6)] for larger values of τ (see Table I). Moreover, for hard turbulence in thermal convection, we discover that $q_\tau(x)$ scales as some power of x:

$$q_\tau(x)\sim A|x|^\alpha \quad \text{for large } |x|\,. \tag{13}$$

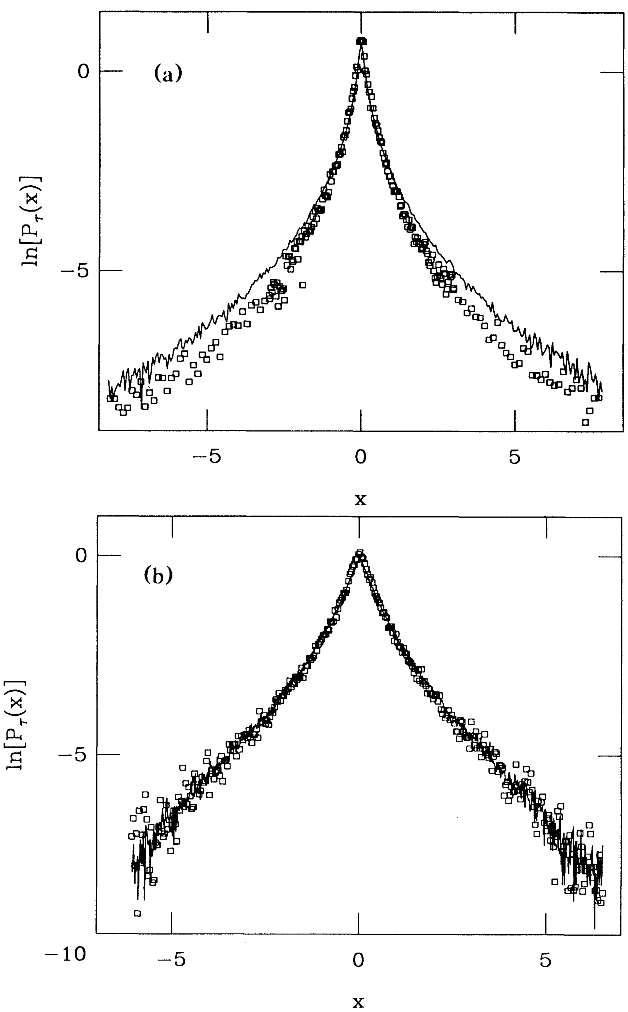

FIG. 2. Similar to Fig. 1 for PDF's of the normalized temperature differences, $P_\tau(x)$, for the convective turbulence data with $Ra=7.3\times10^{10}$. The plotting scheme is the same as Fig. 1 with the squares obtained from $q_\tau(x)$ using Eq. (11). (a) $\tau=1$ and (b) 64 experimental sampling intervals. A noticeable discrepancy between the two PDF's is observed for very short time separations.

Using this result in Eq. (11), we get

$$P_\tau(x) \sim \exp\left[-\frac{|x|^{2-\alpha}}{A(2-\alpha)}\right] \text{ for large } |x| . \quad (14)$$

This is exactly the stretched-exponential behavior that was found before [5]. We check that the values of $2-\alpha$ are consistent with those of β [even for the shortest time separations which demonstrates that the form of Eq. (11) does work well for large $|x|$].

Hence, we have discovered two PDF relations of the Sinai-Yakhot type, one for the PDF of temperature fluctuations [Eq. (5)], and the other for the PDF's of temperature differences [Eq. (11)], which work well for both convective turbulence data and passive temperature data in which fluctuations are sustained by external forcing. These PDF relations can be obtained by generalizing a statistical relation of the fluctuation-dissipation form [Eq. (2)], which is derived [6] only for decaying fluctuations. This suggests that there is a universality in turbulence. We anticipate that similar PDF relations will hold for other scalar fields, like the concentration field (of the dye used as a marker for flow) in wakes of cylinder and turbulent jets [20]. It would be interesting to check whether this is true. With these PDF relations, we can have a classification of different turbulent states according to the different functional forms of q or q_τ. For example, a constant q gives a Gaussian PDF, a power-law behavior of q [$q(x) \sim x^\alpha$ with $0<\alpha<2$] gives a stretched-exponential PDF of $\sim\exp(-|x|^\beta)$ with $\beta=2-\alpha$, while a quadratic q ($q \sim x^2$) gives an algebraic PDF of $\sim|x|^{-p}$ with $p>0$, which is related to the hyperbolic intermittency observed in atmosphere dynamics [21].

I thank A. Libchaber and K. R. Sreenivasan for providing me with their data. It is a pleasure to thank L. Kadanoff for his constant encouragement. I have benefited from many discussions with I. W. Yu. This research was supported by the University of Chicago Material Research Laboratory and in part by the National Science Foundation Grant No. PHY89-04035.

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