On-Off Intermittency: A Mechanism for Bursting

N. Platt
Naval Surface Warfare Center, Silver Spring, Maryland 20903-5000

E. A. Spiegel
Department of Astronomy, Columbia University, New York, New York 10027

C. Tresser
IBM T. J. Watson Laboratories, Yorktown Heights, New York 10598
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On-off intermittency is an aperiodic switching between static, or laminar, behavior and chaotic bursts of oscillation. It can be generated by systems having an unstable invariant (or quasi-invariant) manifold, within which is found a suitable attractor. We clarify the roles of such attractors in producing intermittency, provide examples, and relate them to previous work.

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The simplest chaotic systems follow similar trajectories over and over again but they never exactly repeat. This behavior has long been recognized in celestial mechanics [1], but only in recent times have simple models for such aperiodic oscillatory behavior proliferated. More than this, the time dependence of a variable in a chaotic system can lead to signals with a variety of distinct forms. These range from weak aperiodic modulations of a periodic signal to apparently random switching amongst qualitatively different kinds of oscillations.

The latter behavior is called *intermittency,* probably after the usage of this word in fluid dynamics [2]. In fluid turbulence, the term was introduced to describe signals from probes in fluids that alternated between flat portions and bursty ones, interpreted as laminar and turbulent states of the fluid. We shall say that a process producing this form of intermittency, switching abruptly from extended periods of stasis to bursts of large variation, manifests *on-off* intermittency.

A model of intermittency in terms of simple dynamical systems was given by Pomeau and Manneville (PM) [3]. In their discrete time model, a system spends a long time near a weakly unstable fixed point, or a *quasifixed* point, whose image is not far from the point itself. With the introduction of an aperiodic recurrence mechanism that turns trajectories back toward this unstable fixed point, an intermittent signal is produced. Pomeau and Manneville also proposed a classification for various types of intermittency corresponding to different modes of instability of the fixed point.

The fixed point in the PM model corresponds to a periodic orbit in the continuous time system that they implicitly describe. Hence their intermittency is generally not of the on-off type, for continuous systems. For a simple example of on-off behavior, we can parallel their model with a differential equation admitting a critical point that loses stability with the tuning of a parameter. More generally, we may use any other weakly unstable, invariant objects representing states near to which the system will tend to spend long times. In fact, the objects that organize the behavior do not even have to be invariant. It is enough that they be *quasi-invariant* in the systems that enter their neighborhoods and remain there for a long time. Intermittent systems can be constructed around either invariant or quasi-invariant objects. We shall speak only of the former for brevity, though the latter could serve as well and have been used in previous models of intermittency. An example can be constructed from the unstable invariant object devised by Grebogi, Ott, and Yorke [4], with the introduction of a reinjection mechanism into their model, though the signal produced in this way would again generally not be of the on-off kind.

On-off intermittency occurs when the unstable object lies in the hyperplane $x_1 = \cdots = x_K = 0$ of the phase space where the coordinates $x_1, \ldots, x_N$ with $K < N \leq \infty$ are suitably chosen. Though many systems are capable of producing on-off signals, they may far less often be detected since the "suitable" variables may not arise naturally, nor be discovered easily. Good coordinates are likely to be natural in many real problems where a small set of variables is observable but a larger number of "hidden variables" is believed to be implicated. The solar cycle involves turbulent convection; stock market prices are influenced by various economic factors; in the wild, populations of certain species are environmentally influenced. In these cases, "crashes" are seen—sunspots rarely occurred in Newton's time and species may come close to extinction and yet survive.

In these examples, the codimension $N-K$ is quite large and it is not clear whether the evolution occurring in the complementary space should be characterized as deterministic or random. It is a feature of the mechanism we propose that this distinction does not matter very much for the shape of the observed signals. This insensitivity may at first surprise those aware of Takens' theorem [5] ensuring the possibility of estimating the dimension of phase space from a time signal. However, an essential assumption of this theorem is that the variable observed is generic. In our construction, the observables lie in a hypersurface, so this assumption is not satisfied.

Let $X=(x_1,\ldots,x_K)$ represent the coordinates in the K-dimensional hyperplane and let $Y=(x_{K+1},\ldots,x_N)$. The differential equations describing the evolution of the entire system take the form

$$\dot{X}=F(X,Y,\mu_0)\,, \tag{1}$$

$$\dot{Y}=G(X,Y,\nu_0)\,, \tag{2}$$

where μ_0 and ν_0 are sets of (bare) parameters.

In the simplest version of the mechanism, we have two essential ingredients: (a) that the hyperplane $X=0$ be invariant under the evolution described by (1) and (2) and (b) that there exist orbits entering and leaving every sufficiently small neighborhood of the hyperplane. For simplicity, at first, we furthermore suppose that the dynamics of Y is independent of the observable X. Mathematicians would then say that the evolution equation has a *skew product* structure [6]. Skew products, in this sense, are often used to construct delicate examples and counterexamples [6,7], though it may be a quadrivial problem to learn whether this is essential to the purpose. In the case of ordinary statistical thermodynamics, the convenience of a skew product structure is achieved by assuming that the heat reservoir is infinite and so indifferent to what the subsystem of interest does. In our case, the skew product structure plays a largely pedagogical role and it will become clear that a dependence of Y on X will not necessarily change our conclusions.

A skew product structure provides a clear picture of the dynamics: First of all, it permits us to arrange that (a) is satisfied. Next, suppose that (1) [or (3a) below] admits a strange attractor $\mathcal{A}$ in the hyperplane $X=0$ and that not only is (b) satisfied, but that there exist orbits entering and leaving every sufficiently small neighborhood of $\mathcal{A}$ in the full N-dimensional space. Then, simply if the motion in the X direction is bounded, there exists an invariant bounded region containing $\mathcal{A}$ in which the dynamics has the same degree of Y complexity as that of $\mathcal{A}$ itself. That is, the orthogonal projection of every orbit onto the hyperplane $X=0$ converges to $\mathcal{A}$ as time goes to infinity.

The crux of our proposal is the interpretation of Y as a modifier of the parameters controlling the stability of the observed rest state of X. In the special case of a skew product structure, the X in (2) is suppressed, so that this interpretation is manifest. Indeed, in this case, the evolution equations may be written as

$$\dot{X}=\hat{F}(X,\mu(t))\,, \tag{3}$$

$$\dot{Y}(t)=G(Y,\nu_0)\,, \tag{4}$$

with

$$\mu(t)=M(\mu_0,Y(t))\,. \tag{5}$$

Now assume that the rest state $X=0$ of the equation

$$\dot{X}=\hat{F}(X,\mu_0) \tag{3a}$$

becomes unstable when μ_0 exceeds a certain value μ_c. If the dynamics is such that $Y(t)$ explores both of the regions in which μ goes above and below μ_c and if the time intervals spent in those regions are suitable, then the signal will have the on-off form. In the extreme case $N=K$, when μ is fixed at a constant value only slightly greater than μ_c, we recover the Pomeau-Manneville mechanism [3]. In special circumstances like this, we may hope to compute critical exponents describing the relative times spent in the regular phase as μ_0 is varied. But, in the general situation, each case needs a separate analysis and there is no obvious universality.

These considerations explain the behavior seen in the following example [8] with $K=2$ and $N=5$, which may contain a dependence of Y on X, according to the choice of the parameters:

$$\begin{aligned}
\dot{x}_1&=x_2\,,\\
\dot{x}_2&=-x_1^3-2x_1x_3+x_1x_5-\mu_{01}x_2\,,\\
\dot{x}_3&=x_4\,,\\
\dot{x}_4&=-x_3^3-\nu_{01}x_1^2+x_5x_3-\nu_{02}x_4\,,\\
\dot{x}_5&=-\nu_{03}x_5-\nu_{04}x_1^2-\nu_{05}(x_3^2-1)\,.
\end{aligned} \tag{6}$$

The hyperplane $X=0$ is clearly invariant. When we set $x_1=0$, $x_2=0$, we recover a third order system, which is a version of the Lorenz equations [9]. So, with appropriate choice of parameters, $\mathcal{A}$ is a Lorenz attractor. A sample of the behavior of the variable x_1 is given in Fig. 1. For the parameters of Fig. 1(a), we do not have the skew product structure. This is brought out in Figs. 2(a) and 2(b), respectively, showing the difference between the x_3-x_5 projections of two trajectories, with initial conditions in and out of the hyperplane $X=0$ for that set of parameters. By contrast, for Fig. 1(b) we choose $\nu_{01}=\nu_{04}=0$ to restore the skew product structure.

As usual in such problems, we turn to discrete time systems to facilitate the study of long stretches of orbits. These are indispensable to the investigation of subtle issues such as the distinction between deterministic and stochastic systems. In order to have a simple means of comparing different systems, we need to be able to assure ourselves that the hidden variables in the various cases have comparable statistics. When the skew product structure is available, we know how to do this for many examples.

We write a discrete counterpart of (3)-(5) as

$$X(n+1)=\hat{F}(X(n),\mu(n))\,, \tag{7}$$

$$Y(n+1)=G(Y(n),\nu_0)\,, \tag{8}$$

with

$$\mu(n+1)=M(\mu_0,Y(n))\,. \tag{9}$$

It is convenient to put (9) in the form

$$\mu(n+1)=M(\mu_0,\tilde{Y}(n))\,, \tag{9a}$$

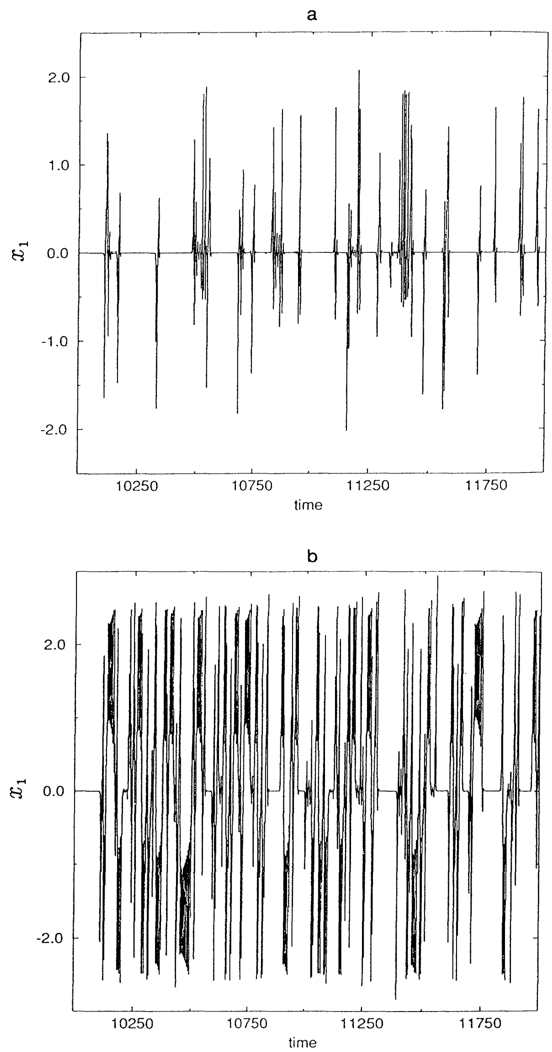

FIG. 1. The time evolution of x_1 from Eqs. (6) with $\mu_{01}=1.815$, $\nu_{02}=1.815$, $\nu_{03}=0.44$, $\nu_{05}=2.86$. In (a), $\nu_{01}=1.0$, $\nu_{04}=2.86$ while in (b), $\nu_{01}=\nu_{04}=0$. The latter choice gives skew product structure to the example.

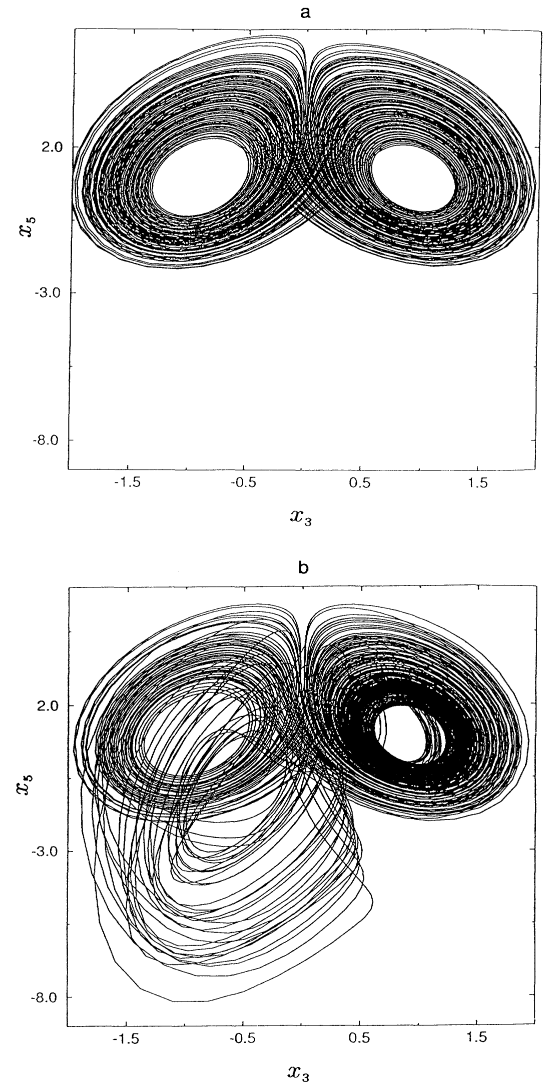

FIG. 2. Two portions of trajectories orthogonally projected onto the x_3-x_5 plane of Eq. (6) with the same parameter values as in Fig. 1(a). In (a), the initial condition lies in the subspace $x_1=x_2=0$ while, in (b), the initial condition is not in this subspace.

where $\tilde{Y}(n)=(x_{K+1}(n),\ldots,x_M(n))$ and $M\leq N$. This permits us to include among the possible specifications of the hidden variables the case of a stochastic background interpreted as an infinite N with an M that in some cases may be inferred from the system.

Consider the example

$$\hat{F}(X,\mu)=\begin{cases}\mu X \text{ if } X\leq \frac{1}{4}, \\ \frac{1}{3}\mu(1-X) \text{ otherwise,}\end{cases} \tag{10a}$$

so that $K=M=1$. Here, μ is determined by

$$\mu=\mu(n+1)=aY(n)\,. \tag{10b}$$

We consider two versions of this example: (A) Y is a deterministic process with $N=1$, namely,

$$Y(n+1)=2Y(n)\bmod 1\,; \tag{11}$$

(B) Y is white noise with support in [0,1], so that $N=\infty$. Thus, in the two versions, the variable x_2 is uniformly distributed in [0,1]. Figure 3 shows some results obtained for (A) and for (B).

The examples shown here were not at all atypical and, for them, it is virtually impossible to tell which one is noise and which one is deterministic, or even that their provenances are of such different character. It will be interesting to see if specialists in quantifying chaos can devise objective means to distinguish between the two situations. Whether that will be feasible or not, the kinds of

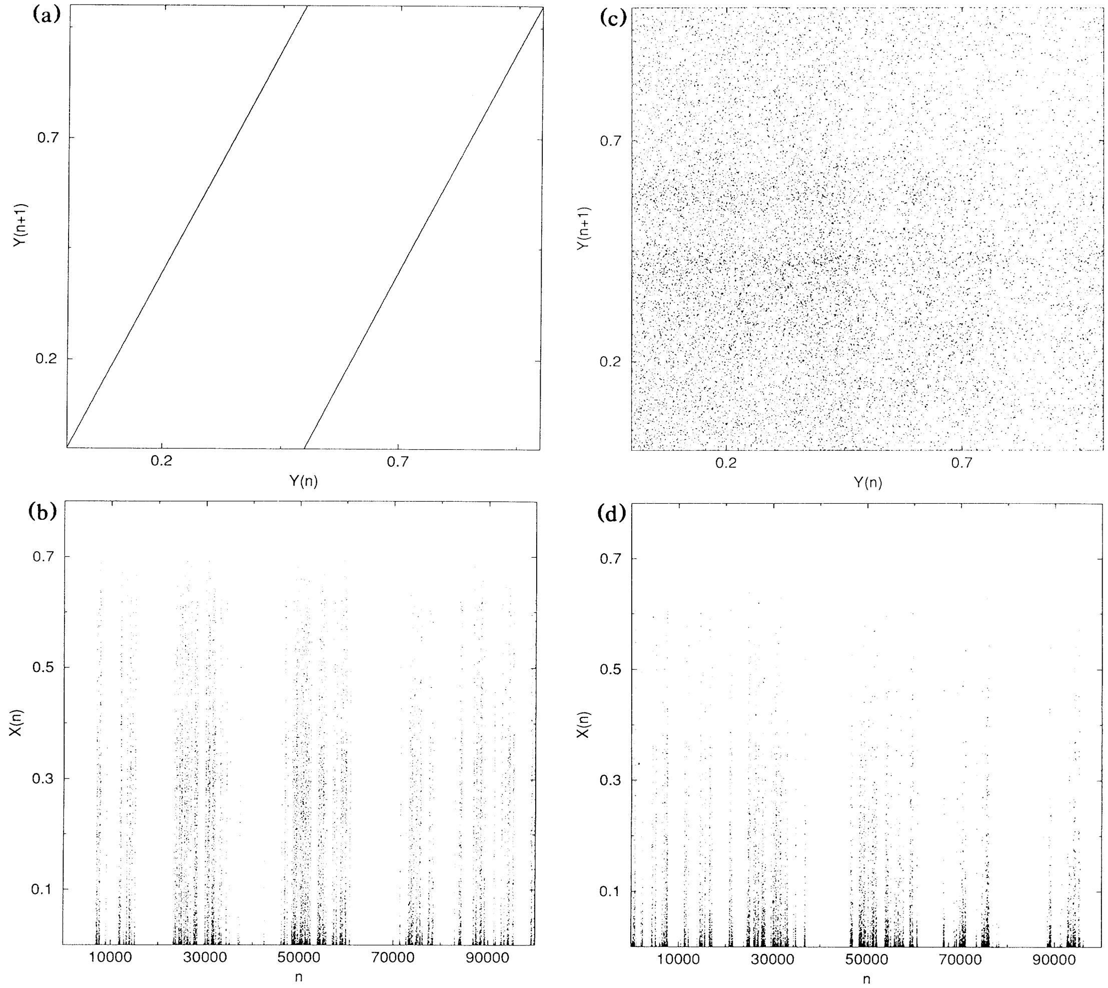

FIG. 3. Illustrating the example of (10). (a) and (c) show $Y(n+1)$ as a function of $Y(n)$. (b) and (d) show the evolution of $X(n)$ with n. Panels (a) and (b) refer to case (A) $Y(n+1)=2Y(n)\mathrm{mod}1$; (c) and (d) correspond to case (B) with $Y(n)$ given by a random number generator with uniform distribution in [0,1].

structures we have sketched here form the basis for a wide class of chaotic, intermittent systems that may help in modeling and understanding a variety of physical systems.

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