Sandpile Models with Dynamically Varying Critical Slopes

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Sandpile models with randomized updating rules are studied. The randomness is not quenched, but set dynamically by the dissipation events (avalanches) in the system. Relaxation from fixed updating rules is here a consequence of medium anisotropy and motivated by the behavior of various driven physical systems. A one-dimensional sandpile model in which the critical slope at a site varies as the process proceeds displays self-organized criticality. The model allows one to build in an internal relaxation mechanism separated from the external driving flux and underlying the noise of the process.

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Bak and co-workers have proposed "self-organized criticality" as a framework to understand the dynamics of driven, dissipative systems [1]. Attention is focused on the dissipation of the energy that is fed into the system, in particular on the statistics of dissipation events ("avalanches"). The inherent dissipation dynamics *drives the systems into* a "critical state" with no characteristic length scale, i.e., with power-law behavior of measured quantities. The hope was that sandpile models might explain the frequent occurrence of both $1/f$ noise (through the dynamics of the process) and self-similar (fractal) spatial structures (through the spatial extension of dissipation events) [1]. Reference [2] reviews various numerical models that display self-organized criticality, in the sense that the avalanche size distributions and related quantities are power laws. However, the Bak model and related models do not and cannot have $1/f$ noise; they have a $1/f^2$ spectrum [3-5], whereas $1/f$ noise can be found at finite driving rates [2]. Diverse processes including earthquakes, turbulence, and economics have been proposed to possess self-organized criticality, but experimental verification is still lacking for most of the candidates [2]. Further studies on both real and idealized experimental systems are clearly needed.

In this Letter we propose that the updating rules should be active parts of the models. Our main argument is that *the updating rules are not distinct from the state of the system, but are rather intrinsic properties determined by its structure, which changes as the process evolves.* In the models the rules are always local, the neighborhood configuration determines what to do next, but these configurations change continuously. Dynamically changing rules have been discussed before but not studied in detail [5-7]. Attempts to incorporate inertial effects in sandpile models utilize dynamically changing rules [8]. The changing rules in our model are a consequence of the medium through which the process propagates, or more precisely, *a consequence of anisotropy in the structures generated in the medium by the process.* Several general questions motivate this study. First, these features may be important when designing new model experiments with possible self-organized criticality. Second, we argue that the noise of these models should be generated by some internal mechanism of the process itself, rather than by an external source. Third, we draw attention to a possible separation of an internal relaxation mechanism from the external driving force.

These general ideas are investigated through a simple numerical model in one dimension (1D). The 1D Bak model did not show critical behavior. A number of 1D sandpile models with nontrivial dynamics have been proposed [9]. Here the distribution of avalanche sizes or durations have a quite complicated behavior indicating that the microscopic rules give rise to new, mesoscopic length scales [10]. Also in our model there will be buildups and subsequent partial discharges of mass and thus a nontrivial dynamics. However, our algorithm does not generate any mesoscopic length scale. Perhaps the most interesting feature of our model is that an internal relaxation mechanism can be incorporated and studied separately.

These models are based on a linear array of cells labeled by i, where $i=1,2,\ldots,L$, and an integer variable $h(i)$ assigned to each of them; see Fig. 1. Consider the "height" $h(i)$ as the number of grains in the column at site i. A sandpile model is defined by specifying how the system is perturbed (new grains added), how steep a slope the system is able to sustain, and how configurations with too steep slopes should evolve. Conventionally, there is a wall at $i=0$ and an edge where grains drop off at $i=L+1$. In Bak's model [1] a grain of sand is added on a randomly chosen site i, $h(i) \rightarrow h(i)+1$. Then all the

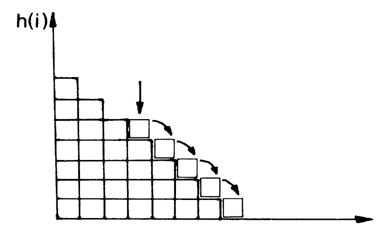

FIG. 1. A 1D sandpile, i.e., a strip of columns of sand. The dynamics of such a system in its critical state (Bak model) is also indicated: Any added grain will be transported through the entire system and will drop off at the right edge.

slopes $\sigma(i) \equiv h(i) - h(i+1)$ are checked. If a slope $\sigma(i) > \sigma_c$, where σ_c is some constant, one grain is moved to the right, $h(i) \to h(i) - 1$, $h(i+1) \to h(i+1) + 1$. This relaxation rule is applied until all slopes $\sigma(i) \le \sigma_c$, before a new grain is added. Figure 1 shows this system in its critical state where all $\sigma(i) = \sigma_c$. Addition of a grain on a site i will render $\sigma(i) > \sigma_c$, so one grain falls over to site $i+1$, and then $\sigma(i+1) > \sigma_c$, etc. The added unit will be transported to the right edge and leave the system unchanged as it drops off. For many physical systems such a response where a "signal" (an added grain) is transported unchanged through the entire system in unrealistic. The Bak model has the same type of response in higher dimensions. The nontrivial dynamics found for $d > 1$ is attributed to a geometrical effect in a perfectly ordered system of identical units; see Fig. 1. Toner studied effects of disorder in grain size and position [11]. However, the disorder was quenched; as the process evolved a new grain of identical size replaced the old one at precisely the same position at the surface of the pile. We find that disorder has a dramatic influence on the *dynamics* of these systems. As a result of the transport process the grains in the surface layer are constantly exchanged. If there are variations in grain size, shape, or surface properties, the configurations may vary locally. But since some configurations are more stable than others, the tolerated (critical) slope may vary along the surface. In preliminary experiments with a quasi 1D granular system we observe these ever-shifting surface configurations [12].

Before we describe the numerical model we will allude to some relevant experimental situations. Experiments on sandpiles under different conditions have been reported [13]. When real sand (and not glass spheres) is used, local variations in packing can be expected. A complication for direct comparison to sandpile models is the nonlocal phenomena present in all these experiments —inertial effects of moving grains and the tendency for whole portions of a pile to slide. Another process in which the total force felt at each position is determined by local configurations varying with the process is the flow of flux lines [6]. The model is of exclusion-repulsion type. In addition to the driving force each particle is repelled from occupied neighbor sites. Analogous processes are the movement of dislocations and the dynamics of magnetic domain walls [7]. A related process occurs during very slow immiscible fluid-fluid displacement influenced by buoyancy in porous media. The growth of the invading fluid structure is dominated by a single finger which fragments into a chain of blobs [14]. The transport along this fluid structure under continuous injection shows long periods with little activity and sudden large drainage events that alter the blob configuration [15]. Still another process is the flow of water between two parallel and closely spaced inclined plates [16]. There is a complex spatiotemporal behavior of the stream width. Also relevant here is probably the low rate regime (not reported in detail in Ref. [16]) where the stream tends to break up into a chain of drops.

Since many driven systems do not have an input that is distributed over the whole system but rather a flux in through a point or a plane, we always add new grains at the left end in our sandpile model. Further, random deposition represents a complicating additional noise (see below). The critical (allowed) slope may take on values 1 or 2; when the critical state is reached any avalanche *leaves behind* a profile with a sequence of these slope values. Thus nothing is stated about the internal mechanisms of the propagating avalanche (little experimental information is available), only the state it leaves the system in is specified. The random slopes arise from medium anisotropy and the noise (the choice of slope 1 or 2) is thus in principle generated by the process. However, complicated algorithms here will remain speculative until much more experimental knowledge on various 1D systems has been collected. We have drawn the two slope values with equal probabilities and uncorrelated to other properties of the process. The noise is external to the process for both continuous [2,17] and discrete [1] (lies in the random deposition) sandpile models.

The starting configuration of the system is $h(i) = 2L$ (kept fixed throughout the process) and $h(i) = h(i-1) - (1 \text{ or } 2)$ for $i = 2,3,\ldots,L$. When a grain is added an avalanche is always initiated. As it propagates a series of new (metastable, critical) slope values are drawn, one at a time and regardless of the old values,

$$h_{\text{new}}(i) \to h_{\text{new}}(i-1) - (1 \text{ or } 2) ,$$

$$\epsilon \to \epsilon + [h_{\text{old}}(i) - h_{\text{new}}(i)] ,$$

for $i = 2,3,\ldots,k$, where k is determined by mass conservation; see Fig. 2. Here ϵ is the mass difference between the old and the new profile. The excess mass ϵ is set equal to 1 as a grain is added and the avalanche initiated. For each i, ϵ is increased (decreased) by the mass amount the new profile is lower (higher) than the old one. The

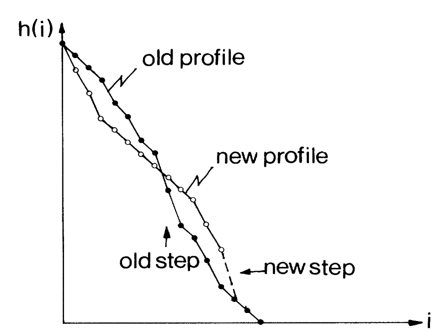

FIG. 2. The change in the profile of the numerical sandpile model generated by one avalanche. Steps formed where the avalanches have been halted are indicated.

avalanche is stopped abruptly when ϵ reaches 0 [it may be necessary to reset $h(k)$ to ensure mass conservation]. The profile remains unchanged to the right of k.

It is unlikely that the old and new profiles coincide at $h(k)$, so steps are formed in the profile; see Fig. 2. We have not introduced any relaxation of them. First, there is no obvious way to implement a relaxation (little experimental information). Second, this is a nontrivial model only due to the steps (see below), and it is therefore preferable that they are evident in the profile. The mass conservation is then the only nonlocal component of the algorithm, and the steps are a direct consequence of it. We also allow a step at the right boundary. The step height at i, $T(i)$, defined as $T(i)=h(i-1)-h(i+1)-3$, is found to behave asymptotically with position s as $T(s)\propto s^{-1/2}$. This is reasonable since the steps represent the separation between segments of random walks. Their scaling behavior ensures that the steps do not introduce any mesoscopic length.

When the algorithm has been applied for a long enough time and a step structure built up, the system reaches a critical state. We find that the avalanche size distribution $D(s)$ is a simple power law, $D(s)\propto s^{1-\tau}$ with τ close to $\frac{7}{2}$; see Fig. 3. Mass conservation [17] has a profound effect on the value of τ. For a different model in which every avalanche conserves mass but moves on a pure random walk profile (so that the model is nonconservative), we measure $\tau\simeq\frac{5}{2}$. This is reasonable since the increment of the excess mass performs a random walk, and the probability for the avalanche to proceed scales with size as the probability for return to origin, $P(s)\propto s^{-\alpha}$, $\alpha=\frac{1}{2}$, i.e., $\tau=\alpha+2=\frac{5}{2}$. In comparison, the steps in the main model efficiently restrain the avalanches and make large ones less probable. The energy dissipation (loss of potential energy) in our main model has a $1/f^2$ power spectrum.

"Fluctuation phenomena are 'the tip of the iceberg' revealing the existence, behind even the most quiescent appearing macroscopic states, of an underlying world of agitated ever changing microscopic processes [18]." To improve sandpile models as tools for understanding fluctuations and responses we believe it is important to consider models where the flux (a tunable external parameter) is separated from the internal relaxation mechanism (an intrinsic property of the system), as in some experimental systems [14–16]. The dispersive nature of our algorithm allows a slightly modified model to be defined towards this end. An avalanche with excess mass $\epsilon=0$ is now allowed to proceed, and is only halted when a move resulting in negative ϵ is attempted. A series of these "zero slides" (started without grain addition) constitutes an "agitated microscopic process" which attempts to transfer mass to the right in the pile. The transport properties of the system now depend on the rate $\gamma=$(number of grains added)/(number of zero slides). Complex correlations arise, the flow of, say, a certain mass amount in the middle of the pile will be very sensitive to γ, and processes with new time scales are generated, as for some experimental systems [14,15]. Here only some indicative results on the transport of a single pulse will be given. A system was driven to its critical state by a $\gamma=1$ flux, the flux was turned off, and the system given a mass pulse by setting $h(j)=h(1)$ for $j=2,3,\ldots,n$. We then recorded the transport of this extra mass (with $\gamma=0$). We used the number of zero slides necessary for transporting all the pulse mass out of a system of size L as a measure for time t. This pulse transport is subdiffusive (filled symbols in Fig. 4). Even a low γ value (instead of $\gamma=0$) after the pulse addition enhances the transport considerably (open symbols in Fig. 4). The significant scatter after the substantial averaging is a consequence of large underlying variations in the total mass of the system in the critical

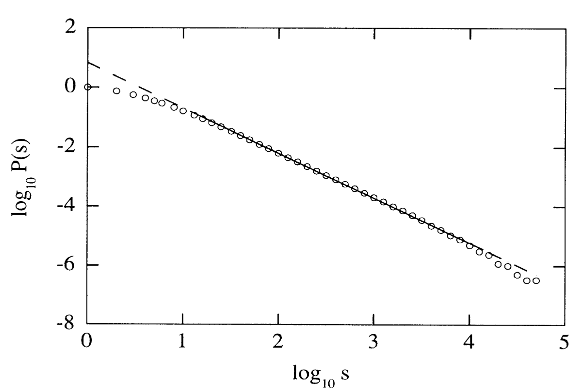

FIG. 3. A log-log plot of the probability $P(s)$ to have avalanches larger than s, as a function of avalanche size s. 6×10^6 avalanches were measured and we found $P(s)\propto s^{-\alpha}$, with $\alpha=1.515\pm0.006$. The exponents α and τ are related, $\tau=\alpha+2$.

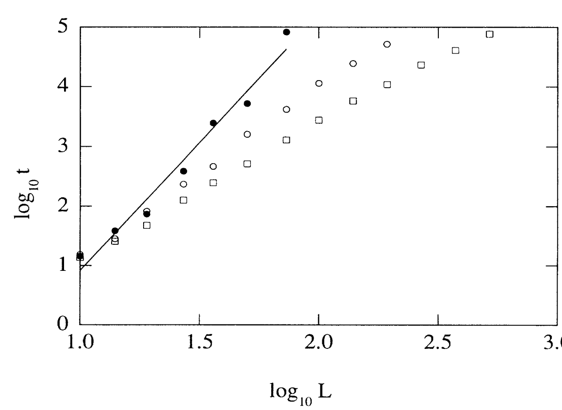

FIG. 4. A log-log plot of the decay time t for a pulse of width $n=4$ at $\gamma=0$ as a function of system size L (●). Here $t\propto L^{\beta}$, with $\beta=4.3\pm0.3$. Decay times with $\gamma=10^{-3}$ (○) and $\gamma=10^{-2}$ (□) are also shown, for which we find $\beta=2.86\pm0.05$ and $\beta=2.25\pm0.03$, respectively. Results from 1000 independent relaxations were averaged for each data point.

state. Generally we expect that a critical state generated by a regular flux will be drained and may break down to an internal relaxation mechanism if the flux drops to a low enough value. For the conventional discrete sandpile models there is no decay independent of the driving force. If the driving flux is turned off the systems maintain their configuration forever. Details on low flux transport and breakdown of the critical state in our model will be given elsewhere.

A more analytical treatment is difficult since the process is strongly affected by its history through the step configuration. Interestingly, one may consider the profile dynamics as the wandering of a front with one end fixed. The wandering consists of specified twistings of front pieces; see Fig. 2. While Hwa and Kardar [17] assume that the profile is on the average flat, we find more dramatic fluctuations.

In conclusion, we have discussed driven systems where the process continuously modifies the medium through which it propagates. In a 1D sandpile model we find that this medium anisotropy leads to a history-dependent process with self-organized criticality. An internal relaxation mechanism separated from the driving flux may be incorporated, and several regimes are expected as the flux is tuned.

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