Frequency Selection and Global Instabilities in Three-Dimensional Weakly Nonparallel Flows

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The temporal evolution of global modes is studied, which are time-harmonic solutions of the linear disturbance equations, subject to homogeneous boundary conditions in all space directions. As basic flow we consider weakly nonparallel three-dimensional shear flows. A necessary condition for the existence of a global mode is the presence of at least two branches of the local dispersion relation and a location where they coalesce. It leads to a mode coupling in a neighborhood of that point. To analyze the mode coupling the uniform asymptotic description of Kravtsov and Ludwig is employed which also yields a general formula for the global eigenfrequency.

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Global instabilities, which develop as a whole on a spatially inhomogeneous flow, are of great interest as they often lead to strong limit-cycle oscillations. In the past, such oscillations have been observed when flow interacts strongly with solid surfaces, such as in edge tone phenomena [1]. More recently, global instability and associated limit cycles have been identified in cases where the dominant role is played by a region of local absolute instability and not by the boundaries. The most notable examples in this category are bluff-body wakes, low-density jets, zonally varying flows in atmosphere, and the dynamo theory of disklike objects [2–13]. The region of absolute instability thereby plays the role of "wave maker" for the entire flow. To date, mostly the temporal instability of *two-dimensional* spatially inhomogeneous flows has been studied [1–11], while in three dimensions only the parallel case has been elucidated [14,15].

Global instabilities in three-dimensional, spatially inhomogeneous shear flows have not yet been tackled theoretically, although they have been the focus of much experimental effort. Cylinder wakes are a case in point where several variations of cylinder diameter, such as taper and steps, have been found to yield generic three-dimensional vortex patterns [16]. The latter very often involve cells of different vortex-shedding frequency along the cylinder span, which is clearly a nonlinear phenomenon. In order to understand the early stages of pattern formation following, say, an initial impulse, we address the problem of linear global modes in fully three-dimensional situations.

For our study of instability waves in a spatially inhomogeneous 3D shear flow, we start for simplicity from the incompressible Euler equations, linearized around a basic flow $\mathbf{U}(y,\mathbf{X})$. To make the problem amenable to analysis we assume a weak spatial dependence of $\mathbf{U}$ on the streamwise and spanwise coordinates which is expressed by introducing the slow variables $X_i=\varepsilon x_i$ $(i=1,2)$ with $\varepsilon \ll 1$, i.e., $\mathbf{U}=\mathbf{U}(y,X_1,X_2)$. Here ε is the ratio of a typical instability wavelength and the minimum of the characteristic evolution lengths of the mean flow in the streamwise and spanwise directions. As a consequence, the mean velocity component U_3 in the cross-stream direction y is small, of order ε. We note that viscous effects do not change the analysis in any essential way as the Reynolds number has to be at least of order ε^{-1} to be compatible with the assumption of weak nonparallelism. Assuming a uniform mean pressure, we can write

$$\mathbf{u}_{\text{tot}}(y,\mathbf{X},t)=\mathbf{U}(y,\mathbf{X})+\mathbf{u}(y,\mathbf{X},t)\,,$$

$$\pi_{\text{tot}}(y,\mathbf{X},t)=\text{const}+\pi(y,\mathbf{X},t)\,,$$

$\mathbf{X}=(X_1,X_2)$. Here $\mathbf{u}$ is the velocity and π is the pressure disturbance, and $\mathbf{u}_{\text{tot}}$, π_{tot} satisfy the Euler equations and the equation of continuity. After eliminating π, one obtains a system of linear partial differential equations (PDE's) for the disturbance velocity,

$$\mathbf{R}(t,y,\mathbf{X},\nabla)\mathbf{u}(y,\mathbf{X},t)=\mathbf{0}\,, \tag{1}$$

where $\mathbf{u}$ is a three-component vector and $\mathbf{R}$ is a 3×3 matrix operator with variable coefficients. Because of the weak nonparallelism we can look for a solution in the form of the geometrical optics expansion (GOE),

$$\mathbf{u}(y,\mathbf{X},t)=\sum_{n=0}\varepsilon^n\mathbf{u}_n(y,\mathbf{X})\exp[\boldsymbol{\theta}(\mathbf{X})/\varepsilon]\times\exp[i(\omega'+\varepsilon\omega_1)t]\,, \tag{2}$$

subject to the boundary conditions

$$|\mathbf{u}(y,\mathbf{X},t)|\to\mathbf{0} \text{ as } |y|,|\mathbf{X}|\to\infty\,. \tag{3}$$

Here $\omega'+\varepsilon\omega_1$ and $\boldsymbol{\theta}$ are complex frequency and phase which are related to the streamwise and spanwise wave numbers p_i by $\mathbf{p}=(p_1,p_2)=\nabla\theta$. Upon expanding $\mathbf{R}$ in powers of ε as in [17] one obtains at order $O(\varepsilon)$ a homogeneous system of ordinary differential equations in y which depend only parametrically on $\mathbf{X}$,

$$\mathbf{R}\mathbf{u}_0=0,\quad |\mathbf{u}_0(y,\mathbf{X})|\to\mathbf{0} \text{ as } |y|\to\infty\,. \tag{4}$$

In the 2D case, after introducing a stream function, one obtains for $\mathbf{R}$ the scalar Rayleigh operator [18]. The solution of (4) can be written as $\mathbf{u}_0=A_0(\mathbf{X})\boldsymbol{\varphi}_0(y;\mathbf{X})$, where $\boldsymbol{\varphi}_0$ is a local eigenmode. This eigenvalue problem leads to a dispersion relation of the form $H(\omega',\mathbf{p},\mathbf{X})=0$. In general, the system supports several modes $\mathbf{p}^m=H^m(\omega',\mathbf{X})$.

We now restrict ourselves to a class of unbounded mean flows, which tend to a uniform flow (with constant velocity) at $|\mathbf{X}|\to\infty$. Hence the dispersion relation at infinity is a nonlinear PDE of the first order with constant coefficients. This implies that the wave numbers are asymptotically identical in all directions as $|\mathbf{X}|\to\infty$. As a consequence the homogeneous boundary conditions (4) cannot be satisfied by any mode $\mathbf{p}^i(\mathbf{X})$ alone and the dispersion relation must have a branch point [2] for a global mode to exist. A branch point in the (X_1,X_2) plane may be found where the necessary conditions for the establishment of the implicit function theorem are not met. Hence, the *necessary* conditions for its existence are $H=H_{\mathbf{p}}=0$ in an anisotropic medium. In the isotropic case, where $H=H(\omega,|\mathbf{p}|,\mathbf{X})$, the condition reduces to $H=H_p=0$. Noting that the dispersion relation is a PDE of the first order but not the first degree for the function $\theta(\mathbf{X})$, we have to require $p_{1X_2}=p_{2X_1}$ in order to have a continuous phase. This equality also assures solvability of the system $H=H_p=0$.

The existence of a branch point, however, violates the assumptions of the high frequency ansatz (2) since it is easily shown by differentiation that p_{1X_i} and P_{2X_i} become infinite at a branch point and the GOE breaks down. In order to have bounded wave number derivatives at a branch point, i.e., a coalescence point of two solution branches, we need to require in the isotropic case

$$H=H_{p_1}=0,\ \ p_{1X_2}=p_{2X_1},\ \ H_{\mathbf{X}}=0\,. \tag{5a}$$

For an anisotropic medium the necessary conditions take the following form:

$$H=H_{\mathbf{p}}=0,\ \ p_{1X_2}=p_{2X_1},\ \ H_{X_1}=0\,. \tag{5b}$$

In (5), $H_{X_1}=0$ follows from the boundedness of p_{jx} at a point where $H_{\mathbf{p}}=0$. Since $p_{1X_2}=p_{2X_1}$, $H_{X_2}=0$ follows automatically. The analogous conditions for the 2D case were derived for the first time by Pierrehumbert [2] and Bar-Sever and Merkine [10].

We now consider the influence of the dispersion relation singularities (5) on the nature of the solution. For simplicity we suppose that the system (5) has one and only one solution, $(\omega',\mathbf{p}',\mathbf{X}')$ or $(\omega',|\mathbf{p}|',\mathbf{X}')$, respectively. In addition, as $\mathbf{X}$ approaches $\mathbf{X}'$, we have to require that $H_x=O(H_p)$. Since one can always find a curve L_0 along which $\exp(\theta/\varepsilon)$ does not decay, one has to require that $A(\mathbf{X})\to 0$ as $|\mathbf{X}|\to\infty$, $\mathbf{X}\in \mathrm{L}_0$, in order to satisfy the boundary conditions (3). Substituting the expansion (2) into (1) one obtains at $O(\varepsilon)$

$$\mathbf{R}\mathbf{u}_1=-(\nabla_{\mathbf{p}}\mathbf{R}\cdot\nabla)\mathbf{u}_0-A\{\omega_1\mathbf{R}_\omega\varphi_0+\mathbf{M}\}\overset{\text{def}}{=}\mathbf{N}\mathbf{u}_0\,, \tag{6}$$

where $\nabla_{\mathbf{p}}\mathbf{R}$ is a block vector with ith component $\partial\mathbf{R}/\partial p_i$, and $\mathbf{M}$ is the block matrix $\mathbf{M}=\frac{1}{2}\sum_{i,j=1,2}R_{p_ip_j}p_{jX_i}\varphi_0$. Applying an orthogonality condition to the last equation one obtains the so-called transport equation for the amplitude $A(\mathbf{X})$,

$$\gamma_1(\mathbf{X})\cdot\nabla A+\gamma_2(\mathbf{X})A=0\,, \tag{7}$$

where $\cdot$ denotes the scalar product. The coefficients γ_i are defined as

$$\gamma_1(\mathbf{X})=\int_{-\infty}^{\infty}\psi_0^*\nabla_{\mathbf{p}}\mathbf{R}\varphi_0\,dy\,,$$
$$\gamma_2(\mathbf{X})=\int_{-\infty}^{\infty}\psi_0^*\{\nabla_{\mathbf{p}}\mathbf{R}\cdot\nabla\varphi_0+\mathbf{M}+\omega_1\mathbf{R}_\omega\varphi_0\}\,dy \tag{8}$$
$$\equiv\gamma_3(\mathbf{X})+\omega_1\gamma_4(\mathbf{X})\,,$$

where ψ_0^* is the solution of the homogeneous adjoint problem and the asterisk denotes Hermitian conjugation. Using identities similar to those derived by Weinberg [17] one can write

$$\gamma_1(\mathbf{X})=-\nabla_{\mathbf{p}}H\int_{-\infty}^{\infty}\psi_0^*\mathbf{R}_\omega\varphi_0\,dy\,. \tag{9}$$

Hence, the coefficient of ∇A is proportional to the group velocity.

To satisfy the homogeneous boundary conditions we had to require the existence of the coalescence point $\mathbf{X}'$, which led to the divergence of $\nabla\mathbf{p}$. We were able to avoid it by choosing $\omega=\omega'$, but now we are faced with another type of singularity which is again brought about by the coalescence of wave vectors associated with different modes. This is a common situation which leads to mode conversion, and a breakdown of the GOE near $\mathbf{X}'$.

Let $\mu_i(i=1,2)$ be the eigenvalues and φ_i the eigenfunctions of the matrix $\mathbf{R}$: $\mathbf{R}\varphi_i=\mu_i\varphi_i$. The normal modes of interest here correspond to the zero eigenvalues $\mu_i=0$ (the respective vectors φ_i are referred to as zero vectors [19]). Since the determinant D of the matrix $\mathbf{R}$ is equal to the product of its eigenvalues, the dispersion relation may be written in the form $D=\mu_1\mu_2=0$. For one of the normal modes labeled by the subscript 1 the eikonal equation reduces to $\mu_1(\mathbf{r},\mathbf{p})=0$. Let $\theta_1(\mathbf{r})$ be the solution of this equation and $\mathbf{p}^1=\nabla\theta_1$. In the leading approximation the field is proportional to φ_1, i.e., $\mathbf{u}_0=A_{01}\varphi_1$, whereas in the first approximation, the field should contain other components. We represent the first approximation field as the sum $\mathbf{u}_1=A_{11}\varphi_1+A_{12}\varphi_2$. Premultiplying the equation for $\mathbf{u}_1$ by ψ_2^* one obtains $A_{12}\sim\mu_2^{-1}\times[\int_{-\infty}^{\infty}\psi_2^*\cdot\mathbf{N}\,dy]|_{p=p^1}$, were $\mathbf{N}$ is defined by (6). With the degeneracy $\mu_2(\mathbf{p}^1)\to\mu_1(\mathbf{p}^1)=0$, one finds $A_{12}\to\infty$. Hence, the GOE leads to an infinite amplitude at the point $(\omega',\mathbf{p}',\mathbf{X}')$ and the location of the singularity is entirely determined by a dispersion relation and by the necessary conditions for the existence of a global mode. The physical reason for the singularity is of course the neglect of mode coupling at $\mathbf{X}'$.

To obtain a finite amplitude at $\mathbf{X}'$ we adopt here the method of "vertical eigenmodes and horizontal rays" [20,21] which is a combination of the Kravtsov and Ludwig [22,23] and the Bretherton [24] approaches (another possible method was described by Budden [25] and Kravtsov [26]):

$$\mathbf{u}=\exp[-i\omega t+\theta(\mathbf{X})/\varepsilon]\sum_{n=0}[\varepsilon^n\mathbf{B}_n(y,\mathbf{X})V(\varepsilon^{-1/2}\rho(\mathbf{X});\lambda)+\varepsilon^{n+1/2}\mathbf{C}_n(y,\mathbf{X})V'(\varepsilon^{-1/2}\rho(\mathbf{X});\lambda)]\,. \quad (10)$$

The function $V(\cdot)$ satisfies the following equation:

$$V''(\lambda)+(\lambda-\tfrac{1}{4}\eta^2)V(\eta)=0,\quad \omega=\omega'+\varepsilon\omega_1+\cdots,\quad \eta=\varepsilon^{-1/2}\rho(\mathbf{X})\,. \quad (11)$$

Here $\rho(X_1,X_2)$ is an unknown function and λ a constant. The choice of scaling for η is motivated by the results of papers [8–11,13]. To satisfy the boundary conditions (3) we are looking for a solution $V(\eta)$ which is quadratically integrable on the infinite interval. It is well known that this is only possible if $V(\eta)$ is an eigenfunction of (11) with eigenvalues $\lambda_n=n+\frac{1}{2}$. The corresponding eigenfunctions can be expressed in terms of the nth Hermite polynomial. For the following, we choose the most unstable eigenvalue $n=0$ [10,11]. Substituting (10) into Eq. (1) and using (11) to eliminate V'' leads to a sequence of equations in ascending order of $\varepsilon^n V$ and $\varepsilon^{n+1/2}V'$. After multiplying the equations obtained at $O(\varepsilon^{n+1/2})$ by $\pm\rho/2$ and adding the $O(\varepsilon^n)$ equations one obtains at leading order

$$\mathbf{R}((\theta_{\mathbf{X}}\pm\tfrac{1}{2}\rho\rho_{\mathbf{X}});(\theta_{\mathbf{X}}\pm\tfrac{1}{2}\rho\rho_{\mathbf{X}}))[\mathbf{B}_0\pm\tfrac{1}{2}\rho\mathbf{C}_0]=\mathbf{0}\,. \quad (12)$$

It follows from Eq. (12) that the functions $s^{\pm}=\theta\pm\frac{1}{4}\rho^2$ satisfy the dispersion relation (5). Once the modified phases $s^{\pm}$ are found, one immediately has $\theta=\frac{1}{2}(s^{+}+s^{-})$; $\frac{1}{2}\rho^2=(s^{+}-s^{-})$. The solvability conditions at order $O(\varepsilon)$ then yield the improved transport equations [23] for $A^{\pm}$,

$$\boldsymbol{\gamma}_1^{\pm}(\mathbf{X})\cdot\nabla A^{\pm}+[\boldsymbol{\gamma}_2^{\pm}(\mathbf{X})\mp(\lambda\pm\tfrac{1}{2})\rho^{-1}\boldsymbol{\gamma}_1^{\pm}(\mathbf{X})\cdot\nabla\rho]A^{\pm}=0\,, \quad (13)$$

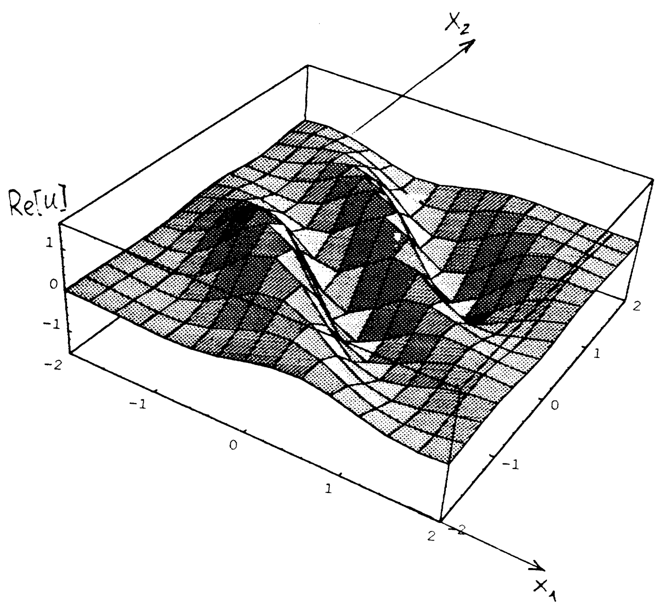

FIG. 1. The real part of $u(\mathbf{X})$ [Eq. (10)] representing the global mode for the particular dispersion relation (see text).

where $\mathbf{B}_0\pm\frac{1}{2}\rho\mathbf{C}_0=A^{\pm}(\mathbf{X})\varphi_0^{\pm}(y;\mathbf{X})$. To obtain expressions for $\boldsymbol{\gamma}_i^{\pm}(\mathbf{X})$ one simply has to add the superscript "$\pm$" to $\boldsymbol{\gamma}_i$, φ_0, and $\boldsymbol{\psi}^*$ in (8).

So far an important feature of the problem, i.e., the zero group velocity at $\mathbf{X}'$ has not been used. Making use of the conditions $H_{\mathbf{p}}=0$, $H_{\mathbf{X}}=0$ one can expand the dispersion relation and consequently the coefficients in Eq. (13) around $\mathbf{X}'$ and find a local solution. To obtain a bounded solution for the envelope $A(\mathbf{X})$ at $\mathbf{X}'$ we have to require

$$\omega_1=-\lambda_0[\gamma_3(\mathbf{X})/\gamma_4(\mathbf{X})]_{\mathbf{X}'}\,. \quad (14)$$

Hence the global mode frequency is solely expressed in terms of the local instability characteristics in the vicinity of the saddle point $\mathbf{X}'$. In more intuitive physical terms, the local oscillator at $\mathbf{X}'$ which experiences the least losses to "mismatched" neighbors assumes the role of "wave maker" for the entire flow. It follows from (11)–(14) that in the vicinity of the absolute instability region [2,7,11] $|\mathbf{X}-\mathbf{X}'|\sim O(\varepsilon^{-1/2})$ the envelope is described by the linear Ginzburg-Landau equation with variable coefficients. This expands the results of papers [2,10,11] to 3D flows.

In order to visualize the spatial structure of a 3D global mode we take an arbitrary dispersion relation $D(\omega,\mathbf{p},\mathbf{X})$ and expand it about $\omega',\mathbf{p}',\mathbf{X}'$. Because of (5), we are left with the quadratic terms as the first approximation to $D(\omega,\mathbf{p},\mathbf{X})$. Solving this PDE for the phase, one obtains the function

$$\rho(\mathbf{X})=(a_1x_1^2+a_2x_1x_2+a_3x_2^2)^{1/2}\,,$$

where the a_i are constants. The global mode $u(\mathbf{X})$ given by (10) is plotted in Fig. 1, for some arbitrary taken coefficients. We note that the dispersion relation can be written in a form of Hamiltonian system. For a conservative system it is known that chaotic motion of rays can occur in the framework of the geometrical optics approach. If this carries over to nonconservative systems, it may open an interesting perspective on the essential differences between "turbulence" in systems with two wave-propagation directions as opposed to one.

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