

## Structure of the Parameter Space of the Hénon Map

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The parameter space of the Hénon map is reported to contain a regular structure-parallel-to-structure sequence of shrimp-shaped robust isoperiodic domains. They appear densely concentrated on a neighborhood along a main  $\alpha$  direction, extending across both orientation-preserving and -reversing domains. There is also a secondary  $\beta$  direction, roughly perpendicular to a very dense "foliation of legs" emanating from the isoperiodic domains. Familiar bifurcation phenomena observed in unimodal maps correspond to particular cuts along the  $\beta$  direction. The  $\alpha$  direction is rich in new phenomena. The topology along  $\alpha$  is conjectured to be typical of bimodal maps.

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A standard way of investigating the dynamics of phenomena common in many scientific disciplines such as physics, engineering, biology, etc. is by modeling them either with differential, difference-differential, or pure difference equations. Such models contain two different classes of quantities: variables and parameters. Familiar concepts such as phase-space, bifurcation diagrams, orbits, attractors, return maps, basins of attraction, manifolds, etc. are nowadays routinely used to extract information of dynamical systems from the variables ruling the dynamics [1,2]. Much valuable information might also be obtained, however, by studying the *dual* space of parameters, particularly regarding certain sets of parameters for which dynamical systems behave similarly with respect to one or several properties of interest such as, for example, periodicity, topology of attractors, basins, return maps, etc. While there are several tools to extract information from the space of variables of dynamical systems, relatively few exist to obtain them from the dual space. The purpose of this Letter is to introduce and apply a tool that was found useful to investigate the space of parameters: isodiagrams. Isodiagrams are simple generalizations of bifurcation diagrams. A bifurcation diagram [2] of a generic dynamical system, say  $\mathbf{x}_{t+1} = \mathbf{f}_\lambda(\mathbf{x}_t)$ , where  $\lambda$  represents collectively one or several parameters, is a plot of one component  $x_t$  as a function of one of the parameters  $\lambda$ . Although bifurcation diagrams are plots of one of the variables, the information that is really of interest in them is the number of different branches that appear as a function of  $\lambda$  or, in other words, the periodicity as a function of  $\lambda$ . If one uses different symbols, e.g., different shadings or colors, to represent different periods, the information contained in the  $x_t$  axis can be compressed into a single line. In other words, by using colors the information contained in 2D bifurcation diagrams can be fully compacted into a 1D polychromatic line. The remainder of this Letter shows how this idea may be used to uncover the structure and useful information contained in the parameter space of dynamical systems depending on more than one parameter. As far as the author knows, the

method and the results reported here have never been presented before.

The problem considered is as follows: How do the sets of parameters characterized by a common property, here periodicity, organize themselves in the parameter space of dynamical systems having codimension higher than 1? As a representative example of phenomena also observed in several other systems, this Letter reports results specific for the familiar two-parameter diffeomorphism of Hénon [3],

$$H_{a,b}(x,y) \mapsto (a - x^2 + by, x), \quad (1)$$

obtained by considering the *simultaneous* variation of both parameters involved. The results are a brief summary of the topology that emerges from a discussion of isoperiodic diagrams contained in two long papers that will appear elsewhere and which concentrate on computational [4] and more mathematical aspects [5].

The parameter space of the Hénon map is found to be organized in a very regular way. Almost all stable complex dynamics occur within or close to a set  $\mathcal{V}$  of parameters which has quite sharp and robust borders well approximated by line segments or simple curves over extended ranges. Embedded in  $\mathcal{V}$  one finds complicated pleopodic shrimplike structures as shown in Fig. 1. Every shrimp consists of a main body of periodicity  $k$  plus an infinite succession of adjacent domains having periods  $k \times 2^n$ , i.e., following a period-doubling route to chaos from the main body  $k$ . For example, the structure at the center of Fig. 1 contains a main body of period 8 (represented by the grey shading), a period-16 region (represented by the thin black adjacent region), followed by a full  $8 \times 2^n$  cascade that due to the scale of the picture cannot be seen anymore. Further magnification uncovers shrimps of higher and higher periods in this domain appearing parallel to each other. A striking regularity found is that essentially all isoperiodic shrimps appear aligned along a very particular direction, say,  $\alpha$ . As seen in Fig. 1, the pleopodic structures appear organized in a distinctive structure-parallel-to-structure way, thereby

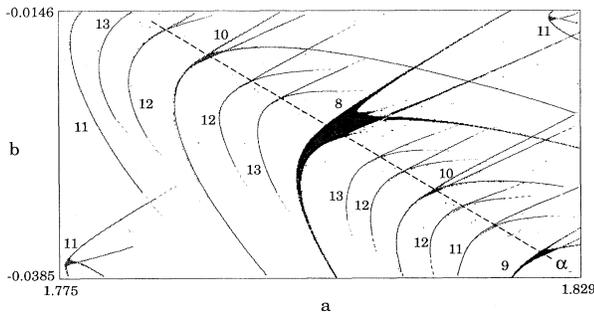


FIG. 1. Isoperiodic shrimps embedded in a chaotic sea along a portion of the direction  $\alpha$ . Numbers refer to main periodicities  $k$ . Note the two “off-diagonal” symmetrically located shrimps of period 11.

defining several further privileged directions. The direction  $\alpha$  is defined by the line  $b = -0.583a + 1.025$ , obtained from a least-squares fit to points located roughly at the center of the main body of many shrimps. A portion of  $\alpha$  can be seen in Fig. 1. A more extended view of this direction appears in Fig. 2. In fact, Fig. 1 is a magnified view of a domain around the 8-shrimp marked in Fig. 2. The intersection of  $\alpha$  with the middle line of crisis  $m$  connecting the vertices at  $b=0$  and  $b=b^*$  occurs at  $b_a \approx -0.059\dots$ . From Fig. 2 one recognizes two sets of “parallel” legs emanating from the 8-shrimp. The circles around both legs going down are used to indicate that two thin legs of periods  $8 \times 2^n$  are born at the shrimp, extending all the way down from it. As one moves away from the central part of the shrimp the legs get thinner and thinner. As might be guessed from the notation, both encircled 5's correspond to legs of periods  $5 \times 2^n$  originating from a shrimp along  $\alpha$  located way above. The same is true for the two encircled  $7 \times 2^n$  legs. Both  $7 \times 2^n$  legs inside squares originate from a different period-7 shrimp. In fact, the dense “foliation” of roughly parallel lines seen to the left and to the right of the period-8 shrimp in Fig. 2 originates from shrimps located almost exclusively along  $\alpha$ , for  $b > 0$ . There is a distinctive direction  $\beta$  “perpendicular” to all legs going down. There is also a further direction, defined by those legs going to the right. From Fig. 2 one recognizes that most of the dynamics known for unimodal maps (exemplified here for the quadratic map along  $b=0$ ) corresponds to “traversing the foliation of legs” coming down from shrimps located in the (orientation-reversing) domain above. The apparently random way with which periodic windows appear embedded in the chaotic domain along  $b=0$  is in fact a consequence of the ordering of shrimps along  $\alpha$ , an important fact that does not seem to have been noticed before.

Figure 3 shows a larger portion of the parameter space while Fig. 4 shows essentially the full domain of parameters corresponding to bounded orbits. Figure 2 is a magnified view of the region  $b^* < b < 0$  around the crisis line  $m$  in Fig. 3 while Fig. 3 itself is a magnification of a

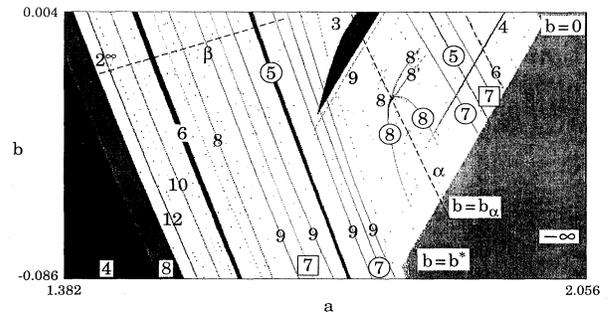


FIG. 2. The direction  $\beta$ . The structure coming down in the upper middle part of the figure is the period-3 window. Phenomena along  $\beta$  essentially coincide with that along  $b=0$ , the “codimension-one” line.

portion of Fig. 4. Figure 3 shows shrimps of main periods 7 ( $B$ ), 5 ( $A$ ), and 7 ( $E$ ) from where the boxed 7, circled 5, and circled  $7 \times 2^n$  legs, respectively, shown in Fig. 2 originate. Shrimps of all higher periods  $k$  can be found in  $\mathcal{V}$  by zooming further and further in finer domains around  $\alpha$ . Such structures, however, get very small quickly. In particular, note that the 8-shrimp of Fig. 1 is the only one still discernible on the scale of Fig. 2. Figure 3 also contains two salient features: a “nose” with a tip at  $(2,0)$ , formed by the intersection of the  $u$  and  $m$  lines, and an “antinose” with a tip at  $(a^*, b^*) \approx (1.8219\dots, -0.0824\dots)$ , the intersection of the  $m$  and  $l$  lines, where degenerate tangencies occur.

The set  $\mathcal{F}$  of pairs  $(a,b)$  for which orbits of  $(x_0, y_0)$  remain bounded to finite attractors is the union of two sets: (i) the set  $\mathcal{P}$ , containing many topologically parallel, not unique, mildly overlapping, robust, isoperiodic shrimplike subsets  $P_k$  of main periodicity  $k$  and secondary adjacent periodicities  $k \times 2^n$ ,  $n=1,2,3,\dots$ , and (ii) the set  $\mathcal{C}$ , containing all chaotic orbits. Sitting on the  $2^\infty$  accumulation border of the big  $1 \times 2^n$  subset is the domain  $\mathcal{V}$ , delimited schematically by the shading roughly at the center of Fig. 3(b).  $\mathcal{V}$  is the union of a chaotic background, including most of the set  $\mathcal{C}$ , and the majority of the autosimilar isoperiodic subsets  $P_k$  described above. Similar sets  $\mathcal{V}$  appearing sometimes as a collection of nonconnected domains were found for several typical physical systems that we investigated, including two ODEs [6]. Therefore we conjecture similar domains  $\mathcal{V}$  containing an infinite number of complex isoperiodic structures ordered in particular ways to be a characteristic topological feature present in the parameter space of physical systems of codimension higher than 1. The shrimp-shape of the  $P_k$ , although occurring frequently in the systems investigated, is not “universal.” The specific ordering with which the several  $P_k$  appear embedded in  $\mathcal{V}$  is, however, highly symmetrical and is the same for large classes of physical systems. At the many borders of the shrimps, small inaccuracies (noise) in choosing parameters are enough to drastically change final behaviors.

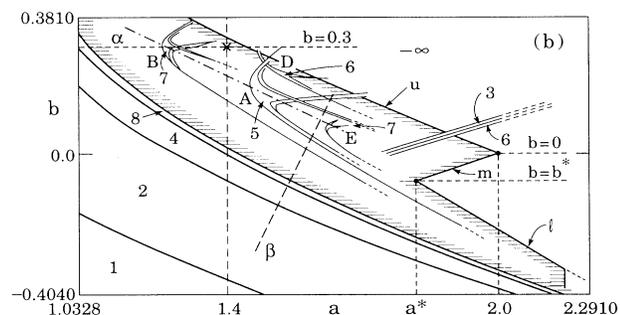
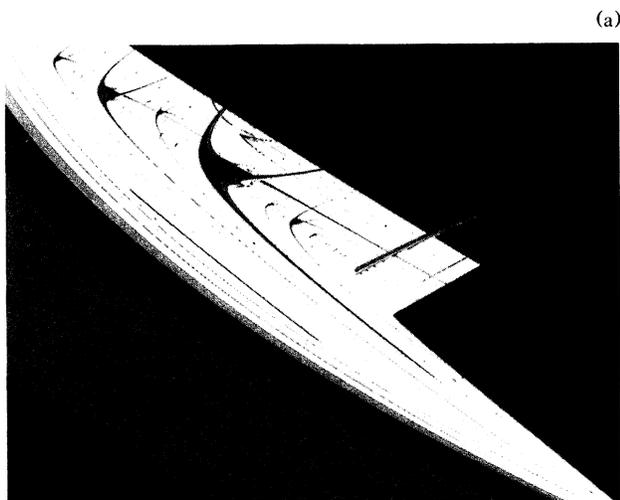


FIG. 3. (a) The *Via Caotica*  $\mathcal{V}$ , delimited by the white region; (b) schematic view showing the sharp vertices at  $(2,0)$  and  $(a^*, b^*)$ , the  $a$  direction with the structures-parallel-to-structures and the  $\beta$  line, signaling the “foliation” of legs (see text). The main period of shrimp  $A$  is 5, of  $B$  and  $E$  is 7, and of  $D$  is 6.  $-\infty$  refers to parameters for which generic orbits are not bounded. Similar colors denote domains of similar periodicity.

Note that by following different routes in the space of parameters it is possible to see many “new” routes to chaos.

Shrimps can be found over extremely wide parameter domains, stretching from orientation-preserving to orientation-reversing domains. Most of them consist of four main thin legs made of an infinite number of continuous and smooth segments resulting from intersections with legs from other shrimps as seen from the figures. At the relatively small intersections the periodicity of both legs is stable and the one that “dominates” on the figures depends on the initial conditions used to follow the orbits (multistability). The dotted appearance of some legs in the figures reflects only a limitation of the printing device used. The  $a$  direction is valid for  $0.94 \leq a \leq 1.86$  which covers the range  $-0.059 \leq b \leq 0.48$ , containing all of the  $u$  line and almost the entire  $m$  line. All shrimps appear densely confined to two strips roughly parallel to the  $a$  line, roughly delimited by the line  $u$  and an inner curve

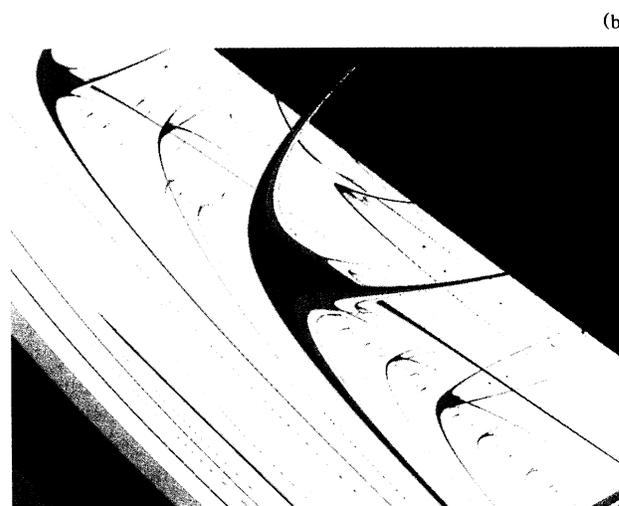
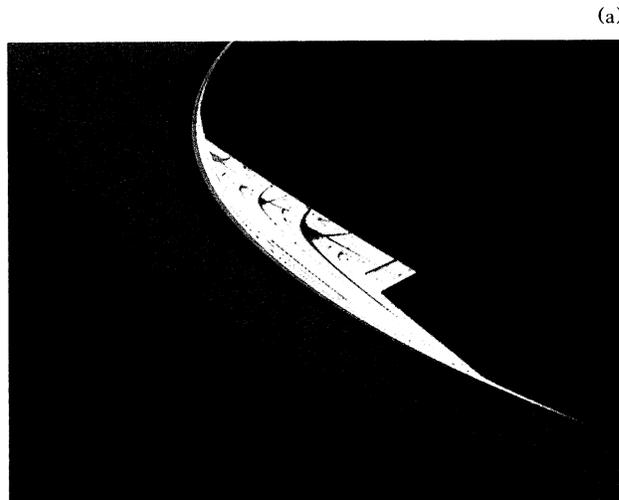


FIG. 4. (a) The domain  $0 \leq a \leq 3$  (horizontal axis)  $-1 < b < 1$  (vertical axis). (b) Magnification of the domain  $1.2 \leq a \leq 1.8$  and  $0.0 \leq b \leq 0.32$ . Colors and symbols are as in Fig. 3.

located at about the same distance between  $a$  and  $u$ , but to the left and “parallel” to  $a$ . While shrimps are predominantly located in the  $b > 0$  region, it is possible to find an infinity of them between the nose ( $b=0$ ) and the antinose ( $b=b^*$ ), i.e., for  $b < 0$ . There is no evidence of shrimps below  $b^*$ . To the left of the region containing the  $P_k$  shrimps one finds the direction  $\beta$ . By moving parallel to  $\beta$  one finds the smallest possible variation of the bifurcation pattern over extended parameter domains. Bifurcation diagrams along  $\beta$ , in contrast to what happens elsewhere, preserve monotonically their structure and relative widths, reflecting the way in which the parallel legs align themselves in that region and along  $a$ . The experience already accumulated from studies along  $\beta$  (particularly along  $b=0$ ) has been frequently used as a

guide to “guess” what happens when  $b \neq 0$ . However, as shown here, the “big action” occurs along  $\alpha$ . The dynamics along  $\alpha$  determines and slaves that observed along  $\beta$ , not the contrary. As seen from Figs. 2 and 3, the domain  $\mathcal{V}$  has quite sharp borders between bounded attractors and the attractor at  $-\infty$ , borders which are well approximated by simple curves over extended ranges of parameters. The *upper* border indicated in Fig. 3 is roughly contained between (0.973, 0.560) and (2, 0), the *middle* border between (2, 0) and (1.8219, -0.0824), and the *lower* border between (1.8219, -0.0824) and (2.328, -0.441). There are other sharp discontinuities at (0.949, 0.677) and (2.328, -0.441) similar to those at (2, 0) and  $(a^*, b^*)$ .

By properly changing both parameters simultaneously it is possible to navigate over extended domains without ever changing the periodicity of the attractor. It is also possible to find paths along which the periodicity does not change or changes just a small number of times and which connect several shrimps. In fact, it is possible to recognize a complex *network* of paths which allow one to move around *shrimps of all periodicities* without ever needing to cross the vast surrounding chaotic sea. Such a network is of interest for “controlling” chaotic physical systems [7], here through *macroscopic* changes of parameters along *stable* orbits. These two characteristics are very important for experiments. The diagrams presented show which parameter(s) to choose in order to change from, say, a given initial stable periodic behavior to a final *desired* one on a single move. They show how to move to avoid chaos, if this is at premium. Few and precise moves are interesting features for controlling behavior of physical systems.

Isoperiodic diagrams help to identify and unify behaviors typical of higher-codimensional systems that are not easy to recognize from bifurcation diagrams. An example involves reports in the literature of some “new routes” to chaos occurring in families of maps (especially some maps with discontinuities) and obtained by considering the dynamics along particular one-parameter cuts of higher codimensional systems. Such interesting reports provide a glimpse of the complicated way in which “generalized” shrimps occur in high codimension and that might be conveniently explained with isoperiodic diagrams. The diagrams are also helpful in the study of non-Markovian processes simulated via discrete dynamical systems [8].

This study generates a number of interesting questions, such as the following: Although found for several systems, how generic is the topology reported here? Why do shrimps tend to appear essentially aligned along specific and simple curves? What is the proper “Sarkovski-like ordering” [2] for full shrimps, not intervals? What is the generic pattern behind the highly symmetrical *ordering*

of shrimps evident in Fig. 1? What is the mathematical reason for it? Why do  $2^n$  cascades always appear oriented in the same way? Is it possible to apply/generalize results of Milnor and Thurston [9] for full shrimps? Is a two-symbol grammar enough to uniquely label all trajectories of the Hénon invariant set through primary homoclinic tangencies over all the space of parameters? Is this the simplest method of characterizing the dual space? Are there “generalized” structures embedded in multimodal higher-codimensional systems, which “collapse to shrimps” in codimension two? Why do shrimps exist on both sides of the  $b=0$  line, the line thought to separate markedly different physical behaviors? All this and certainly more remains, however, to be investigated.

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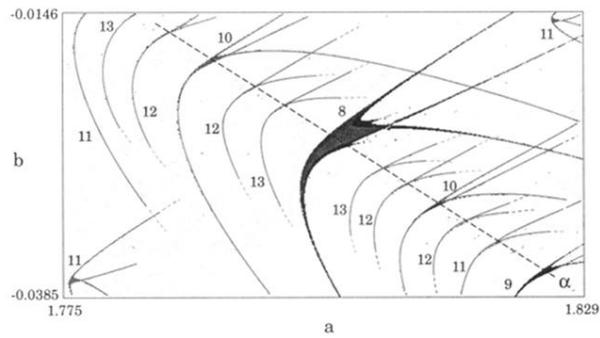


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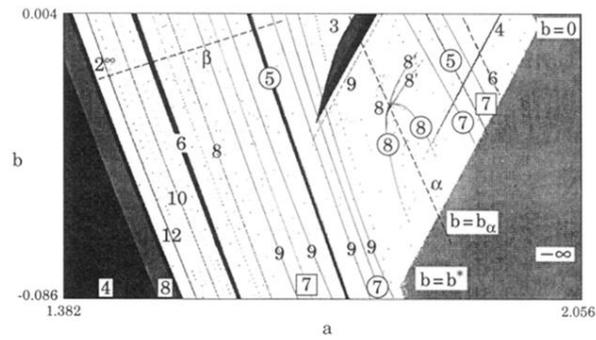


FIG. 2. The direction  $\beta$ . The structure coming down in the upper middle part of the figure is the period-3 window. Phenomena along  $\beta$  essentially coincide with that along  $b=0$ , the “codimension-one” line.

(a)

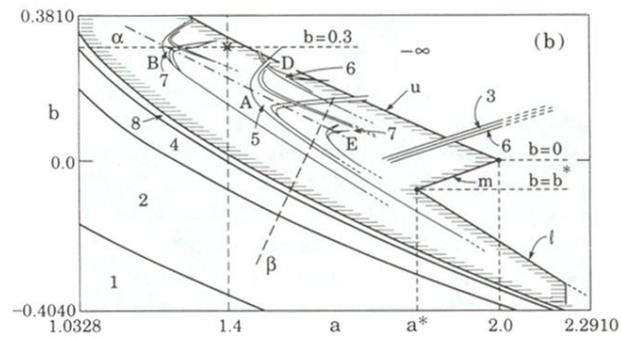
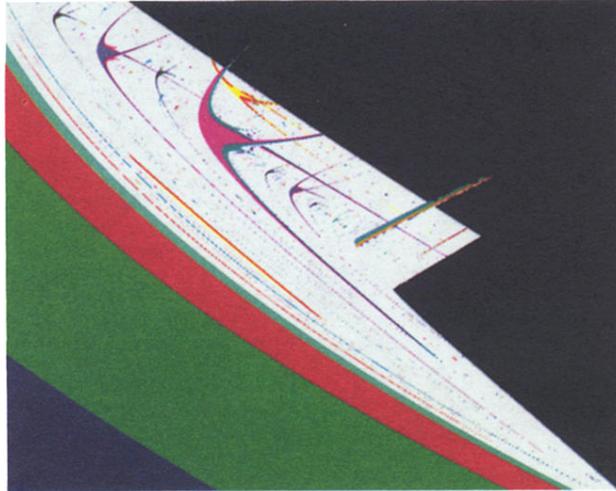


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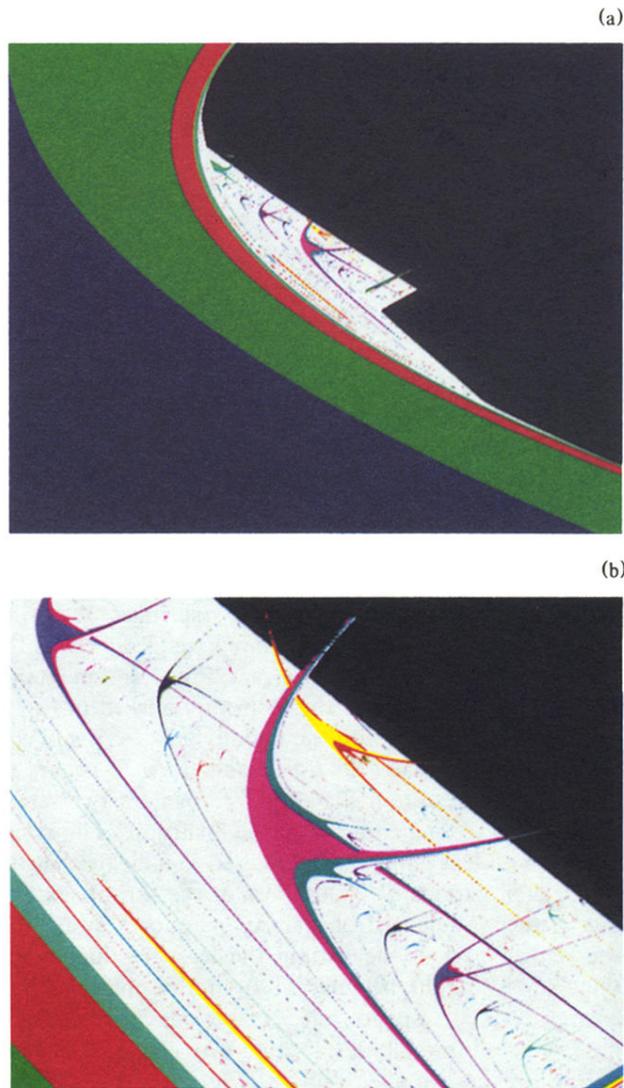


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