One-Loop Corrections to Five-Gluon Amplitudes

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We present the one-loop helicity amplitudes with five external gluons. The computation employs string-based methods, new techniques for performing tensor integrals, and improvements in the spinor helicity method.

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Calculations beyond the leading order in quantum chromodynamics are important in refining our understanding of known physics in present-day and future collider experiments, such as the Tevatron or the SSC and LHC. In jet physics, next-to-leading-order calculations are important in curing several deficiencies of their leading-order counterparts: the strong spurious dependence on the renormalization scale, the lack of sensitivity to the jet resolution parameters, namely, the minimum transverse energy and the jet cone size, and the absence of a warning about dangerous infrared logarithms. The one-loop corrections to matrix elements for $2 \rightarrow 2$ processes in QCD, a key ingredient of the next-to-leadingorder calculations of inclusive-jet and two-jet cross sections and distributions, were computed by Ellis and Sexton [1]. To go beyond these cross sections, whether to higher orders for two-jet cross sections, or to next-toleading order for three-jet cross sections and distributions, requires the computation of the one-loop corrections to the $2 \rightarrow 3$ matrix elements. At hadron colliders, the QCD coupling α_s , and the manner of its running, can be extracted from purely hadronic processes by comparing three-jet production to two-jet production, at various center-of-mass energies. The presence of infrared logarithms in both of these quantities means that this cannot be done sensibly unless both quantities are known to next-to-leading order.

We present here the one-loop matrix elements with five external gluons, which are the hardest part of a $2 \rightarrow 3$ calculation if a traditional diagrammatic method is used. We have performed the calculation using the string-based methods developed in Ref. [2] as more efficient tools for one-loop calculations with external gluons. The rules presented there were derived by taking the infinitetension limit of an appropriately constructed heterotic string amplitude. The structure of the rules can also be understood in conventional field theory [3], and the application of such methods to a calculation such as the present one does not require any knowledge of string theory. (It turns out that it is possible to construct a set

of rules yielding more compact integral representation of gluon amplitudes at intermediate stages than would emerge from a straightforward application of the rules in Ref. [2]. This alternate set of string-based rules will be discussed elsewhere.)

In the string-based method, one first decomposes the n-gluon amplitude, depending on the external momenta, helicities, and color indices k_i , λ_i , and a_i , into sums over certain permutations of color factors, times partial amplitudes, in analogy to the helicity [4,5] and color [6] decomposition of tree amplitudes. At one-loop order in an $SU(N)$ theory, one must also sum over the different spins J of the internal particles; this takes the following form when all internal particles transform as color adjoints:

$$
\mathcal{A}_n(\{k,\lambda_i,a_i\}) = \sum_{J} n_J \sum_{c=1}^{\lfloor n/2 \rfloor + 1} \sum_{\sigma \in S_n/S_{n,c}} \text{Gr}_{n,c}(\sigma) A_{n,c}^{[J]}(\sigma) ,
$$
\n(1)

where

$$
Gr_{n;1}(1) = N Tr(T^{a_1} \cdots T^{a_n}),
$$

\n
$$
Gr_{n;c}(1) = Tr(T^{a_1} \cdots T^{a_{c-1}}) Tr(T^{a_c} \cdots T^{a_n}),
$$

 S_n is the set of all permutations of *n* objects, and $S_{n,c}$ is the subset leaving the trace structure $Gr_{n,c}$ invariant. The T^a are the set of Hermitian traceless $N \times N$ matrices, normalized so that $Tr(T^aT^b) = \delta^{ab}$. For internal particles in the fundamental $(N+\overline{N})$ representation, only the single-trace color structure $(c=1)$ is present, and it is smaller by a factor of N . We take in each case a spin- J particle with two states: gauge bosons, Weyl fermions, and complex scalars.

The objects one calculates are the partial amplitudes $A_{n;c}^{[J]}$, which depend only on the external momenta and helicities. For the five-point function, there is only one independent partial amplitude for each configuration of external helicities; $A_{5,2}$ and $A_{5,3}$ are related to the adjoint contributions to $A_{5,1}$ via decoupling equations [7].

The string-based method meshes well with the spinor helicity representation for the polarization vectors [4,5],

tensor

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 $c(i,j,m,n) = 4i\epsilon_{\mu\nu\rho\sigma}k_i^{\mu}k_j^{\nu}k_m^{\rho}k_n^{\sigma}$

 $\langle i l \rangle [l k]$ 2s_{il}s_{kl}

 $\left\{ \frac{i}{j}\right\}$ $\left\{ \frac{s_{il}s_{jk}+s_{kl}s_{ij}-s_{ik}s_{jl}-\varepsilon(i,j,k,l)}{\varepsilon(j,k,j,k)}\right\}$

using, e.g., methods used in Ref. [8]. In this way spinor products can be eliminated from any expression, apart

For massless five-point kinematics, such an expression can then be written as a rational function in the five kinematic variables $\{\beta_1,\beta_2^*,\beta_3,\beta_4^*,\beta_5\}$ (or any cyclic permuta-

cally [10]. The corresponding one-loop amplitudes are

then free of infrared divergences. The remaining ampli-

ization. The computation of these helicity amplitudes thus requires the knowledge of five-point loop integrals in

 $A_{n;c}^{[1]} = -A_{n;c}^{[1/2]} = A_{n;c}^{[0]}$. (This holds true for the partial amplitudes whether or not the theory as a whole is super-
symmetric.) Indeed, in the string-based method, these

It suffices to calculate the ratios

from an overall prefactor.

tion of this set), where

 $= [ij] \langle jm \rangle [mn] \langle ni \rangle - \langle ij \rangle [jm] \langle mn \rangle [ni]$.

(2)

(3)

which provides an efficient method for extracting the essential gauge-invariant information in an on-shell amplitude. This method yields expressions written in terms of spinor products $\langle ij \rangle$ and [ij], which are defined by $\langle ij \rangle = \overline{\psi}_{-}(k_{i})\psi_{+}(k_{j}),$ [ij] $= \overline{\psi}_{+}(k_{i})\psi_{-}(k_{j}),$ where $\psi_{\lambda}(k)$ is a massless Weyl spinor with momentum k and helicity λ . Up to a phase, $\langle ij \rangle$ and $[ij]$ are square roots of the Lorentz products $s_{ij} = (k_i + k_j)^2$. Unfortunately, the relations—momentum conservation and the Schouten relations—momentum conservation and the Schouten identity—between different forms of a given expression are nonlinear, which makes it hard to give a canonical form for such expressions, or equivalently makes it hard to simplify complicated expressions. However, one can evaluate "phase-invariant" combinations of spinor prod ucts in terms of s_{ij} and contractions of the Levi-Civita

$$
\beta_i = [i, i+1] \langle i+1, i+2 \rangle [i+2, i+3] \langle i+3, i \rangle \left(- \prod_{j=1}^5 s_{j,j+1} \right)
$$

The only independent Levi-Civita contraction is given by $\varepsilon(1,2,3,4)/(-\prod_{j=1}^{5} s_{j,j+1})^{1}$ $+\beta_3\beta_5+\beta_4^*\beta_1)/\beta_3 = \beta_i - \beta_i^*$ dent Lorentz products by $s_{i,i+1} = -1/(\beta_i + \beta_{i+1}^*) (\beta_{i+2})$ tudes are infrared divergent; for practical purposes these $+\beta_{i+3}^*$). Simplification of rational functions in β_i is divergences must be regulated using dime $+\beta_{i+3}^*$). Simplification of rational functions in β_i is divergences must be regulated using dimensional regular-
straightforward. ization. The computation of these helicity amplitudes

The β_i variables are related to the variables γ_i and $\hat{\Delta}_5$ thus requires the ed in Ref. [9] to perform pentagon integrals, via $\beta_i^{(*)}$ $D=4-2\epsilon$ [9,11]. used in Ref. [9] to perform pentagon integrals, via $\beta_i^{(*)}$ $D=4-2\epsilon$ [9,11]. $= -(\gamma_{i+2} \pm \hat{\Delta}_5^{1/2})/2$. Indeed, the derivative approach to For the finite helicity amplitudes, supersymmetric idenevaluating tensor integrals [9], when applied to the penta- tities [12] imply that the contributions of particles of gon integrands encountered in the five-gluon calculation, different spin circulating around the loop are related, and expressed in terms of the appropriate set of β_i variables, allows one to significantly reduce the degree and size of the Feynman parameter polynomials in the integrand. **identifies hold for the integrands of each diagram.** The

At tree level, certain helicity amplitudes vanish identi- amplitudes are

$$
A_{5;1}^{[1]}(1^+,2^+,3^+,4^+,5^+) = \frac{i}{96\pi^2} \frac{s_{12}s_{23}+s_{23}s_{34}+s_{34}s_{45}+s_{45}s_{51}+s_{51}s_{12}+ \varepsilon(1,2,3,4)}{(12)(23)(34)(45)(51)},
$$

\n
$$
A_{5;1}^{[1]}(1^-,2^+,3^+,4^+,5^+) = \frac{i}{48\pi^2} \frac{1}{[12](23)(34)(45)[51]}
$$

\n
$$
\times \left[(s_{23}+s_{34}+s_{45})[25]^2 - [24](43)[35][25] - \frac{[12][15]}{\langle 12\rangle\langle 15\rangle} \left[\langle 12\rangle^2\langle 13\rangle^2 \frac{[23]}{\langle 23\rangle} + \langle 13\rangle^2\langle 14\rangle^2 \frac{[34]}{\langle 34\rangle} + \langle 14\rangle^2\langle 15\rangle^2 \frac{[45]}{\langle 45\rangle} \right] \right].
$$
\n(4)

In order to present the results for the remaining, infrared-divergent amplitudes in a compact form, it is helpful to define the following functions:

$$
L_0(r) = \frac{\ln(r)}{1-r}, \quad L_1(r) = \frac{\ln(r) + 1 - r}{(1-r)^2}, \quad L_2(r) = \frac{\ln(r) - (r - 1/r)/2}{(1-r)^3},
$$
\n
$$
L_{S_1}(r_1, r_2) = \frac{1}{(1-r_1 - r_2)^2} \left[\text{Li}_2(1-r_1) + \text{Li}_2(1-r_2) + \ln r_1 \ln r_2 - \frac{\pi^2}{6} + (1-r_1 - r_2) [L_0(r_1) + L_0(r_2)] \right],
$$
\n(5)

where $Li₂$ is the dilogarithm; a prefactor,

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$$
c_{\Gamma} = \frac{(4\pi)^{\epsilon}}{16\pi^2} \frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \,, \tag{6}
$$

a universal function,

$$
V^g = -\frac{1}{\epsilon^2} \sum_{j=1}^5 \left(\frac{\mu^2}{-s_{j,j+1}} \right)^{\epsilon} + \sum_{j=1}^5 \ln \left(\frac{-s_{j,j+1}}{-s_{j+1,j+2}} \right) \ln \left(\frac{-s_{j+2,j-2}}{-s_{j-2,j-1}} \right) + \frac{5}{6} \pi^2 - \frac{\delta_R}{3}, \tag{7}
$$

the following functions for the $(1^-, 2^-, 3^+, 4^+, 5^+)$ helicity configuration,

$$
V^f = -\frac{5}{2\epsilon} - \frac{1}{2} \left[\ln \left(\frac{\mu^2}{-s_{23}} \right) + \ln \left(\frac{\mu^2}{-s_{51}} \right) \right] - 2, \quad V^s = -\frac{1}{3} V^f + \frac{2}{9},
$$

\n
$$
F^f = -\frac{1}{2} \frac{\langle 12 \rangle^2 (\langle 23 \rangle [34] \langle 41 \rangle + \langle 24 \rangle [45] \langle 51 \rangle) \langle 26 \langle -s_{23} \rangle - s_{51} \rangle}{\langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 51 \rangle} ,
$$

\n
$$
F^s = -\frac{1}{3} \frac{[34] \langle 41 \rangle \langle 24 \rangle [45] \langle (23) [34] \langle 41 \rangle + \langle 24 \rangle [45] \langle 51 \rangle) \langle 26 \langle -s_{23} \rangle - s_{51} \rangle}{\langle 34 \rangle \langle 45 \rangle} - \frac{1}{3} F^f
$$

\n
$$
-\frac{1}{3} \frac{\langle 35 \rangle [35]^3}{[12] [23] \langle 34 \rangle \langle 45 \rangle [51]} + \frac{1}{3} \frac{\langle 12 \rangle [35]^2}{[23] \langle 34 \rangle \langle 45 \rangle [51]} + \frac{1}{6} \frac{\langle 12 \rangle [34] \langle 41 \rangle \langle 24 \rangle [45]}{s_{23} \langle 34 \rangle \langle 45 \rangle s_{51}},
$$

\n(8)

and the corresponding ones for the $(1^-,2^+,3^-,4^+,5^+)$ helicity configuration:

$$
V^{f} = -\frac{5}{2\epsilon} - \frac{1}{2} \left[\ln \left(\frac{\mu^{2}}{-s_{34}} \right) + \ln \left(\frac{\mu^{2}}{-s_{51}} \right) \right] - 2, \quad V^{s} = -\frac{1}{3} V^{f} + \frac{2}{9},
$$
\n
$$
F^{f} = -\frac{\langle 13\rangle^{2}\langle 41\rangle[24]^{2}}{\langle 45\rangle\langle 51\rangle} \frac{Ls_{1}(-s_{23}/-s_{51}, -s_{34}/-s_{51})}{s_{51}^{2}} + \frac{\langle 13\rangle^{2}\langle 53\rangle[25]^{2}}{\langle 34\rangle\langle 45\rangle} \frac{Ls_{1}(-s_{12}/-s_{34}, -s_{51}/-s_{34})}{s_{54}^{2}} - \frac{1}{2} \frac{\langle 13\rangle^{3}(\langle 15\rangle[52] \langle 23\rangle - \langle 34\rangle[42] \langle 21\rangle) \left[L_{0}(-s_{34}/-s_{51}) \right]}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle} - s_{51}, -s_{34}/-s_{51} \rangle + L_{1}(-s_{23}/-s_{51}) + L_{1}(-s_{34}/-s_{51})
$$
\n
$$
F^{s} = -\frac{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle^{2}[24]^{2}}{\langle 54\rangle\langle 51\rangle\langle 24\rangle^{2}} \frac{2Ls_{1}(-s_{23}/-s_{51}, -s_{34}/-s_{51}) + L_{1}(-s_{12}/-s_{34}) + L_{1}(-s_{13}/-s_{34})}{s_{54}^{2}}
$$
\n
$$
+ \frac{2}{3} \frac{\langle 23\rangle^{2}\langle 41\rangle^{3}[24]^{3}}{\langle 43\rangle\langle 25\rangle^{2}} \frac{L_{2}(-s_{12}/-s_{34}, -s_{51}/-s_{34}) + L_{1}(-s_{12}/-s_{34}) + L_{2}(-s_{34}/-s_{51})}{s_{54}^{3}}
$$
\n
$$
\times \left[\frac{1
$$

For positive values of s_{ij} , the logarithms and dilogarithms \vdash develop imaginary parts according to the prescription $s_{ij} \rightarrow s_{ij} + i\epsilon$. We also remind the reader of the tree amplitudes,

$$
A_5^{\text{tree}}(1^-,2^-,3^+,4^+,5^+) = i\langle 12\rangle^4/(\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle)
$$

and

$$
A_5^{\text{tree}}(1^-, 2^+, 3^-, 4^+, 5^+) = i \langle 13 \rangle^4 / (\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 51 \rangle).
$$

In terms of these functions, the $\overline{\text{MS}}$ (modified minimal subtraction scheme) renormalized amplitudes are

$$
A_{5;1}^{[0]} = c_{\Gamma}(V^s A_s^{\text{tree}} + iF^s) ,
$$

\n
$$
A_{5;1}^{[1/2]} = -c_{\Gamma}[(V^f + V^s) A_s^{\text{tree}} + i(F^f + F^s)],
$$

\n
$$
A_{5;1}^{[1]} = c_{\Gamma}[(V^g + 4V^f + V^s) A_s^{\text{tree}} + i(4F^f + F^s)].
$$
\n(10)

The rest of the helicity amplitudes are related by cyclic permutations or complex conjugation to those given above. It is interesting to note that in supersymmetric theories, the V^s and F^s terms cancel out of the final amplitude, and that in $N=4$ supersymmetric theories only the V^g term survives. The separation implied above into $g, f,$ and s pieces arises naturally on a diagram-bydiagram basis within the string-based approach. In this approach the V^g term represents the only calculational difference between the contributions with gluons circulating around the loop, and those with fermions; this term has a particularly simple expression at every intermediate stage of the calculation. The parameter δ_R controls the variant of dimensional regularization scheme [2]: For $\delta_R = 0$, one obtains the four-dimensional helicity scheme, while for $\delta_R = 1$ one obtains the 't Hooft-Veltman scheme.

There are several checks we have applied. We have checked gauge invariance, both by computing amplitudes with longitudinal gluons, verifying that one obtains zero,

and by calculating a helicity amplitude with an alternate choice of spinor-helicity reference momenta, and verifying that the result is unchanged. In addition, the forms given above display manifestly the reflection symmetries expected of the amplitudes, symmetries that are not present in the contributions of the individual diagrams. The amplitudes also have consistent limits as one of the gluon momenta becomes soft, and as two adjacent momenta become collinear.

A next-to-leading order, only the infrared-divergent helicity amplitudes (5)–(10) enter into the construction of a program for physical quantities. In order to construct such a program for three-jet quantities, one must form the interference of the tree amplitude with the loop amplitude; this has the form [7]

$$
\sum_{\text{colors}} [\mathcal{A}_{5}^{*} \mathcal{A}_{5}]_{\text{NLO}} = 2g^{8}N^{4}(N^{2} - 1)
$$
\n
$$
\times \left[\text{Re} \sum_{\sigma \in S_{5}/Z_{5}} A_{5}^{\text{tree*}}(\sigma) A_{5;1}(\sigma) + \frac{1}{N^{2}} \text{Re} \sum_{\rho \in S_{5}/Z_{5}} [A_{5}^{\text{tree*}}(r \cdot \rho) A_{5;1}(\rho) - A_{5}^{\text{tree*}}(\rho) A_{5;1}(r \cdot \rho)] + \frac{2}{N^{2}} \text{Re} \sum_{h \in H_{5}} \sum_{\rho \in P(\S)} A_{5}^{\text{tree*}}(h \cdot \rho) A_{5;3}(\rho) \right],
$$
\n(11)

where r is the permutation (24135), $P(\frac{5}{3})$ is the tenelement set of distinct partitions of five elements into lists of length two and three, and $H_5 = \{(12345), (34125),$ $(31245), (21345), (32145), (34215)$. For QCD with n_f flavors of massless quarks, one substitutes $A_{5;1} \rightarrow A_{5;1}^{11}$
+ $(n_f/N)A_{5;1}^{11/2}$ and $A_{5;3} \rightarrow A_{5;3}^{111}$ into Eq. (11). One must then combine this virtual correction with the singular terms in the $2 \rightarrow 4$ matrix elements arising from the integration over soft and collinear phase space. The Giele-Glover formalism [13] makes use of the color ordering in construction of universal functions representing the results of the soft and collinear integrations, and is the most convenient one for doing so. We have used it to check that the poles in ϵ do cancel as expected.

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