Levinson's Theorem for the Dirac Equation

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Levinson's theorem for the Dirac equation is known in the form of a sum of positive and negative energy phase shifts at zero momentum related to the total number of bound states. In this Letter we prove a stronger version of Levinson's theorem valid for positive and negative energy phase shifts separately. The surprising result is that, in general, the phase shifts for each sign of the energy do not give the number of bound states with the same sign of the energy (in units of π), but instead are related to the number of bound states of a certain Schrödinger equation, which coincides with the Dirac equation at zero momentum.

Consider the reduced radial Schrödinger equation

$$u_{kl}'' - \left[\frac{l(l+1)}{r^2} + 2mV - k^2\right]u_{kl} = 0 , \tag{1}$$

for a scattering state characterized by the reduced radial wave function $u_{kl}(r)$ subject to the boundary conditions

$$u_{kl}(0) = 0 \tag{2}$$

at the origin and

$$u_{kl}(r) \rightarrow 2\sin\left(kr - \frac{\pi l}{2} + \eta_l(k)\right) \tag{3}$$

at infinity ($r \rightarrow \infty$), where $\eta_l(k)$ is the phase shift. Here we assume that the potential $V(r)$ is less singular at the origin than $1/r^2$ and that it vanishes at infinity faster than $1/r$. Levinson's theorem [1] asserts that

$$\eta_l(0) = n_l\pi , \tag{4}$$

and thus establishes a connection between the scattering phase shift $\eta_l(0)$ at threshold (zero momentum), for a given angular momentum l, and the number of bound states n_l of the Schrödinger equation (1). If the Schrödinger equation has a zero-energy solution which vanishes at the origin and is finite at infinity and yet not normalizable (it is called a half-bound state or zero-energy resonance and is possible only if $l = 0$) then, as was first shown by Newton [2], Levinson's theorem is modified to read

$$\eta_0(0) = (n_0 + \tfrac{1}{2})\pi . \tag{5}$$

Levinson's theorem, which turned out to be fairly general [3, 4], is one of the most interesting results in quantum theory. It has many potential applications and has been applied recently in atomic physics [5, 6], in quantum field theories [7, 8], and in solid state physics (where it is known in a modified form under the name Friedel's sum rule [9]).

Consider now the reduced radial Dirac equation

$$u_{1\epsilon\kappa}' + \frac{\kappa}{r}u_{1\epsilon\kappa} - (\epsilon + m - V)u_{2\epsilon\kappa} = 0 ,$$

$$u_{2\epsilon\kappa}' - \frac{\kappa}{r}u_{2\epsilon\kappa} + (\epsilon - m - V)u_{1\epsilon\kappa} = 0 , \tag{6}$$

where ϵ is the energy, $V(r)$ is the time component of a vector potential, and $\kappa = \pm 1, \pm 2, \ldots$. The quantum number κ is the standard parametrization of the total angular momentum $j = |\kappa| - 1/2$ and of the relative orientation between the spin and the orbital angular momentum. The appropriate boundary conditions at the origin and infinity are

$$u_{1\epsilon\kappa}(0) = 0 , \tag{7}$$

$$u_{1\epsilon\kappa}(r) \rightarrow \sqrt{\frac{\epsilon+m}{2\epsilon}}2\sin\left(kr - \frac{\pi l}{2} + \eta_{\epsilon\kappa}(k)\right) ,$$

$$u_{2\epsilon\kappa}(0) = 0 , \tag{8}$$

$$u_{2\epsilon\kappa}(r) \rightarrow \sqrt{\frac{\epsilon+m}{2\epsilon}}\frac{2\kappa_0 k}{\epsilon+m}\sin\left(kr - \frac{\pi\bar{l}}{2} + \eta_{\epsilon\kappa}(k)\right) ,$$

where $k = \pm\sqrt{\epsilon^2 - m^2}$, $\kappa_0 = \kappa/|\kappa|$, $l = |\kappa| - (1 - \kappa_0)/2$, $\bar{l} = l - \kappa_0$, and $\eta_{\epsilon\kappa}(k)$ is the phase shift. To ensure the consistency of (7) and (8) with (6) we assume that $V(r)$ behaves like or less singularly than $1/r$ at the origin and that it vanishes at infinity faster than $1/r$. The first cor-

rect statement of Levinson's theorem for Dirac particles was given by Ma and Ni [10],

$$\eta_{m\kappa}(0) + \eta_{-m\kappa}(0) = \left(N_\kappa^+ + N_\kappa^-\right)\pi, \tag{9}$$

which is valid whenever there is no threshold resonance and

$$\eta_{m\kappa}(0) + \eta_{-m\kappa}(0) = \left(N_\kappa^+ + N_\kappa^-\right)\pi + (-1)^l(\pi/2)[\sin^2\eta_{m\kappa}(0) + \sin^2\eta_{-m\kappa}(0)], \tag{10}$$

which is valid for the case with a threshold resonance (which can appear only in the case $\kappa = \pm 1$; see [4]). Here $\pm m$ is the threshold energy of the Dirac particle, N_κ^+ is the number of positive and N_κ^- the number of negative energy bound states of the Dirac equation (6), and $\eta_{\pm m\kappa}(0)$ are the phase shifts at threshold. Prior to the work of Ma and Ni claims were published [11, 12] stating that Levinson's theorem is valid for positive and negative energies separately and in the same sense as in the nonrelativistic case, i.e., $\eta_{\pm m\kappa}(0) = N_\kappa^\pm\pi$, but later such claims were found incorrect [10]. However, we shall prove in this Letter that in a modified sense these claims are correct and that

$$\eta_{m\kappa}(0) = \begin{cases} n_l^+\pi, & l = 0, 1, \ldots, \\ \left(n_0^+ + \tfrac{1}{2}\right)\pi, & l = 0, \end{cases} \tag{11}$$

$$\eta_{-m\kappa}(0) = \begin{cases} n_{\bar{l}}^-\pi, & \bar{l} = 0, 1, \ldots, \\ \left(n_0^- + \tfrac{1}{2}\right)\pi, & \bar{l} = 0, \end{cases} \tag{12}$$

where n_l^+ and $n_{\bar{l}}^-$ are the numbers of bound state solutions of certain radial Schrödinger equations with the angular momenta l and $\bar{l} = l - \kappa/|\kappa|$. In (11) and (12) the first case refers to a situation without a threshold resonance and the second case to a situation with a threshold resonance. Equations (11) and (12) constitute the stronger version of Levinson's theorem for Dirac particles. Notice that from (9) and (11), (12) it follows that

$$N_\kappa^+ + N_\kappa^- = n_l^+ + n_{\bar{l}}^-, \tag{13}$$

whereas in general $N_\kappa^+ \neq n_l^+$ and $N_\kappa^- \neq n_{\bar{l}}^-$. Below it will be shown that (13) is actually a general statement (independent of Levinson's theorem).

The basic observation, which we need in order to derive (11) and (12), is the fact that one can write the Dirac equation (6) as a set of Schrödinger-like equations. In fact, eliminating $u_{2\epsilon\kappa}$ from (6), we obtain

$$u_{1\epsilon\kappa}'' - \left[\frac{\kappa(\kappa+1)}{r^2} - \frac{\kappa}{r}\frac{V'}{\epsilon+m-V} - V^2 + 2\epsilon V - k^2\right]u_{1\epsilon\kappa} + \frac{V'}{\epsilon+m-V}u_{1\epsilon\kappa}' = 0. \tag{14}$$

Eliminating $u_{1\epsilon\kappa}$ from (6) leads to

$$u_{2\epsilon\kappa}'' - \left[\frac{\kappa(\kappa-1)}{r^2} + \frac{\kappa}{r}\frac{V'}{\epsilon-m-V} - V^2 + 2\epsilon V - k^2\right]u_{2\epsilon\kappa} + \frac{V'}{\epsilon-m-V}u_{2\epsilon\kappa}' = 0. \tag{15}$$

Equations (14) and (15) are equivalent to the Dirac equation (6) provided the boundary conditions (7) and (8) are imposed. Actually it is not necessary to solve (14) and (15) simultaneously. For instance, if (14) is solved for $u_{1\epsilon\kappa}$ one gets $u_{2\epsilon\kappa}$ through the first of Eqs. (6). In order to get rid of the last term in (14) and (15) we introduce the new wave functions $w_{1\epsilon\kappa}$, $w_{2\epsilon\kappa}$ defined through

$$u_{1\epsilon\kappa}(r) = \sqrt{\frac{\epsilon+m-V(r)}{2\epsilon}}w_{1\epsilon\kappa}(r), \tag{16}$$

$$u_{2\epsilon\kappa}(r) = \frac{\epsilon k\kappa}{|\epsilon||k||\kappa|}\sqrt{\frac{\epsilon-m-V(r)}{2\epsilon}}w_{2\epsilon\kappa}(r). \tag{17}$$

Putting these in (14) and (15), we obtain

$$w_{1\epsilon\kappa}'' - \left[\frac{l(l+1)}{r^2} - \frac{\kappa}{r}\frac{V'}{\epsilon+m-V} + \frac{1}{2}\frac{V''}{\epsilon+m-V}\right.$$
$$+\frac{3}{4}\left(\frac{V'}{\epsilon+m-V}\right)^2 - V^2$$
$$\left.+2\epsilon V - k^2\right]w_{1\epsilon\kappa} = 0, \tag{18}$$

$$w_{2\epsilon\kappa}'' - \left[\frac{\bar{l}(\bar{l}+1)}{r^2} + \frac{\kappa}{r}\frac{V'}{\epsilon-m-V} + \frac{1}{2}\frac{V''}{\epsilon-m-V}\right.$$
$$+\frac{3}{4}\left(\frac{V'}{\epsilon-m-V}\right)^2 - V^2$$
$$\left.+2\epsilon V - k^2\right]w_{2\epsilon\kappa} = 0, \tag{19}$$

where we used $\kappa(\kappa+1) = l(l+1)$ and $\kappa(\kappa-1) = \bar{l}(\bar{l}+1)$ which are readily obtained from $l = |\kappa| - (1 - \kappa_0)/2$ and $\bar{l} = l - \kappa_0$, $\kappa_0 = \kappa/|\kappa|$. Equations (18) and (19) are of the Schrödinger type except that the potential depends on the energy. Solving (18) and (19) is equivalent to solving the original Dirac equation (6) provided the boundary conditions at the origin and infinity,

$$w_{1\epsilon\kappa}(0) = 0, \tag{20}$$

$$w_{1\epsilon\kappa}(r) \to 2\sin\left(kr - \frac{\pi l}{2} + \eta_{\epsilon\kappa}(k)\right),$$

$$w_{2\epsilon\kappa}(0) = 0 \,, \tag{21}$$

$$w_{2\epsilon\kappa}(r) \to 2\sin\left(kr - \frac{\pi\bar{l}}{2} + \eta_{\epsilon\kappa}(k)\right) \,,$$

are taken into account. Equations (18) and (19) are useful because they are not coupled and hence the phase shift $\eta_{\epsilon\kappa}(k)$ can be computed using either one of them without reference to the other and for each of the positive and negative energies ϵ separately. Yet, we cannot apply Levinson's theorem to (18) or (19) directly since the potential in these equations depends on the energy, and one can show that in this case the theorem is not valid [4]. However, consider the following equation:

$$w_{kl}^{+''} - \left[\frac{l(l+1)}{r^2} - \frac{\kappa}{r}\frac{V'}{2m-V} + \frac{1}{2}\frac{V''}{2m-V} \right.$$
$$\left. + \frac{3}{4}\left(\frac{V'}{2m-V}\right)^2 - V^2 \right.$$
$$\left. + 2mV - k^2 \right] w_{kl}^{+} = 0 \,, \tag{22}$$

$$w_{k\bar{l}}^{-''} - \left[\frac{\bar{l}(\bar{l}+1)}{r^2} - \frac{\kappa}{r}\frac{V'}{2m+V} - \frac{1}{2}\frac{V''}{2m+V} \right.$$
$$\left. + \frac{3}{4}\left(\frac{V'}{2m+V}\right)^2 - V^2 \right.$$
$$\left. - 2mV - k^2 \right] w_{k\bar{l}}^{-} = 0 \,, \tag{23}$$

which are subject to the boundary conditions

$$w_{kl}^{+}(0) = 0 \,, \tag{24}$$

$$w_{kl}^{+}(r) \to 2\sin\left(kr - \frac{\pi l}{2} + \eta_l^{+}(k)\right) \,,$$

$$w_{k\bar{l}}^{-}(0) = 0 \,, \tag{25}$$

$$w_{k\bar{l}}^{-}(r) \to 2\sin\left(kr - \frac{\pi\bar{l}}{2} + \eta_{\bar{l}}^{-}(k)\right) \,.$$

At threshold $(k = 0)$ (22) and (24) coincide with (18) and (20) for $\epsilon = m$, and similarly (23) and (25) coincide with (19) and (21) for $\epsilon = -m$. Moreover, both sets of equations and boundary conditions are analytical near the threshold. Therefore

$$\eta_l^{+}(0) = \eta_{m\kappa}(0) \tag{26}$$

and

$$\eta_{\bar{l}}^{-}(0) = \eta_{-m,\kappa}(0) \,. \tag{27}$$

Equations (22) and (23) are just usual Schrödinger equations, linear in the energy k^2, so that we can apply Levin-

son's theorem (4) and (5) and obtain the desired equations (11) and (12), where n_l^{+} is the number of bound state solutions $(k^2 < 0)$ of (22) and $n_{\bar{l}}^{-}$ is the number of bound state solutions of (23). Actually one would expect an ambiguity in (26) and in (27), each in terms of an additive integer multiple of π. However, it is easy to see that both integers (say n_1 and n_2) must be zero. This follows from the fact that the simultaneous change of m to $-m$ and κ to $-\kappa$ is a symmetry operation, which implies $n_1 = n_2$. Equations (9) and (13), on the other hand, imply $n_1 = -n_2$, and hence both integers are zero. To prove that (13) is valid generally (independently of Levinson's theorem) we make the following observation. Let us multiply the potential V in (18) and (19) and in (22) and (23) by a coupling constant g. For $g = 0$ there are no bound states and (13) is (trivially) valid. We now change g continuously from 0 to 1 and show that (13) remains valid. Assume that (13) is valid for some value of g. Then, if g is increased, at some point a (half-) bound state will either appear or disappear. This happens simultaneously for (18) and (22), and/or for (19) and (23) and in the same direction, for the corresponding equations are equal at and analytic near $k = 0$. Hence the process of bound states entering or leaving through the point $k = 0$ does not alter the validity of (13). But this is not all. Parallel to this process there is a motion of bound states of the Dirac equation (18) and (19) through the point $\epsilon = 0$ (equivalently $k^2 = -m^2$). This process, however, does not change the total number of bound states since, here, if a positive energy bound state disappears, simultaneously a negative energy bound state appears or vice versa. These two processes are all that happens to the number of bound states, if we drive g from 0 to 1. Consequently, we arrive safely at $g = 1$ with (13) still being valid.

It is an interesting fact to notice that (22) and (23) do not correspond to the usual expansion based on the Foldy-Wouthuysen scheme (see [13]). More details and an application of Levinson's theorem to a nonperturbative approach to quantum electrodynamics will be given in [4].

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[1] N. Levinson, K. Danske Vidensk. Selsk. Mat-fys. Medd. **25**, No. 9 (1949).
[2] R. G. Newton, J. Math. Phys. **1**, 319 (1960).
[3] J. M. Jauch, Helv. Phys. Acta **30**, 143 (1957).
[4] N. Poliatzky, "Normalization of Scattering States, Scattering Phase Shifts and Levinson's Theorem" (to be published).
[5] Z. R. Iwinski, Leonard Rosenberg, and Larry Spruch,

Phys. Rev. Lett. **54**, 1602 (1985).

[6] Z. R. Iwinski, Leonard Rosenberg, and Larry Spruch, Phys. Rev. A **33**, 946 (1986).

[7] R. Blankenbecler and D. Boyanovsky, Physica (Amsterdam) **18D**, 367 (1986).

[8] A. J. Niemi and G. W. Semenoff, Phys. Rev. D **32**, 471 (1985).

[9] J. Friedel, Nuovo Cimento Suppl. **7**, 287 (1958).

[10] Z. Q. Ma and G.-J. Ni, Phys. Rev. D **31**, 1482 (1985); **32**, 2203 (1985); **32**, 2213 (1985).

[11] M.-C. Barthélémy, Ann. Inst. Henri Poincaré **7**, 115 (1967).

[12] G.-J. Ni, Phys. Energ. Fortis Phys. Nucl. **3**, 432–49 (1979).

[13] J. D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics* (McGraw-Hill, New York, 1964).