Integrability of a Globally Coupled Oscillator Array

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We show that a dynamical system of N phase oscillators with global cosine coupling is completely integrable. In particular, we prove that the N-dimensional phase space is foliated by invariant two-dimensional tori, for all $N \geq 3$. Explicit expressions are given for the constants of motion, and for the solitary waves that occur in the continuum limit. Our analysis elucidates the origin of the remarkable phase space structure detected in recent numerical studies of globally coupled arrays of Josephson junctions, lasers, and Ginzburg-Landau oscillators.

Globally coupled oscillators are a particularly simple class of many-body dynamical systems, in which each oscillator is coupled to all the others [1-15]. This type of coupling arises in series arrays of Josephson junctions [1-6], electronic oscillator circuits [7], laser arrays [8], charge-density waves [9], and multimode lasers [10,11].

Recent theoretical work has revealed that globally coupled systems are prone to an enormous degree of neutral stability, at least when the oscillators are identical [3-6,8,12-15]. There are also hints of a remarkable phase space structure. For instance, while studying the dynamics of a series array of N overdamped Josephson junctions driven by a dc-bias current and coupled through a resistive load, Tsang *et al.* [4] found numerical evidence that all solutions were periodic or doubly periodic, for all N and for a wide range of parameters. In geometrical terms, the N-dimensional phase space appeared to be *foliated by invariant two-dimensional tori*, suggesting the existence of $N-2$ constants of motion [16]. Ordinarily one would have expected some chaotic regions in phase space, or in the case of an integrable Hamiltonian system, tori of much higher dimension $N/2$. The puzzle is why the Josephson array has so many apparent constants of motion. There are precedents for systems with $N-2$ constants of motion ("nonholonomic integrable systems" in classical mechanics [17]), but these systems have no sources of dissipation, unlike the overdamped, resistively loaded Josephson array of Ref. [4].

While trying to understand the origin of the conjectured 2-tori, we were led to the system

$$\dot{\theta}_i = \omega + \frac{\varepsilon}{N} \sum_{j=1}^{N} \cos(\theta_j - \theta_i), \quad i = 1, \ldots, N. \tag{1}$$

Equation (1) was obtained [5] from the Josephson array equations via the method of averaging [17], valid in the limit of weakly coupled junctions. Here the angles θ_i are related to the junction phases by a certain nonlinear transformation [5], and ω and ε are the transformed bias current and coupling strength, respectively. Equation (1) also arises as the averaged system for globally coupled van der Pol oscillators with linear conservative coupling [18], and for the globally coupled complex Ginzburg-

Landau equation at the Benjamin-Feir transition [15].

In this Letter we show that the averaged system (1) is completely integrable. Its phase space is foliated by invariant 2-tori for all $N \geq 3$; apparently some remnant of this structure persists in the original Josephson array [19]. By an explicit change of variables we show that the phase differences $\theta_i(t) - \theta_j(t)$ oscillate periodically, and that a Hamiltonian system with 1 degree of freedom governs their dynamics. In the continuum limit the system has solitary waves, which are also given explicitly.

Unlike the Toda lattice and other integrable N-body systems [17], Eq. (1) does not come explicitly from a Hamiltonian. Thus (1) may have theoretical interest, in addition to its applications to oscillator arrays.

By going into a rotating frame and rescaling time, we may reduce (1) to

$$\dot{\theta}_i = \frac{1}{N} \sum_{j=1}^{N} \cos(\theta_j - \theta_i), \quad i = 1, \ldots, N. \tag{2}$$

Note that (2) is effectively $(N-1)$-dimensional dynamically—the system can be split into a mean phase and $N-1$ phase differences. The mean phase is driven by the dynamics of the phase differences, but it does not couple back to them. Hence to show that (2) is completely integrable, it suffices to find $N-2$ independent constants of motion. These have been found by trial and error. Since the verification is unenlightening, we outline the results; see [20] for details.

Let $S_{ij} = \sin[(\theta_i - \theta_j)/2]$ and $I = S_{12}S_{23} \cdots S_{N-1,N}S_{N1}$. Then I is a constant of motion [21], as can be checked by differentiation and trigonometric identities. Because (2) is invariant with respect to all permutations, we can now permute the indices in I to obtain $N!$ constants of motion. Precisely $N-2$ of these are functionally independent (except on a set of θ's of measure zero).

Figure 1 illustrates the integrability for $N = 3$. Instead of considering the entire phase space, we may restrict attention to the prism shown in Fig. 1. This canonical invariant region [3,22] corresponds to a particular ordering of the oscillators, say $\theta_3 \leq \theta_2 \leq \theta_1 \leq \theta_3 + 2\pi$. (The ordering never changes since the oscillators cannot pass each other.) On the bottom and side faces, two θ's are equal;
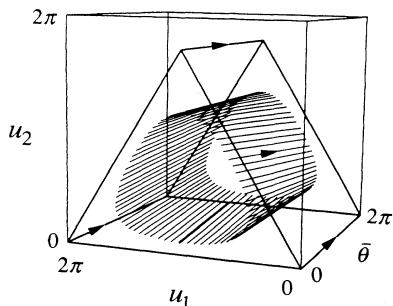
FIG. 1. A typical trajectory of (1) for $N=3$. The transformed coordinates [3,7] are $\bar{\theta} = \frac{1}{3}\sum_j \theta_j \pmod{2\pi}$, $u_1 = \theta_1 - (\theta_2 + \theta_3)/2$, $u_2 = (\sqrt{3}/2)(\theta_2 - \theta_3)$. The trajectory lies on a tube, which is actually a torus since $\bar{\theta}=0$ and $\bar{\theta}=2\pi$ are equivalent.

on the edges extending in the $\bar{\theta}$ direction, all three θ's are equal, corresponding to a solution where all the oscillators run in phase. Finally, there are periodic boundary conditions on the front and back faces, since $\bar{\theta}=0$ and $\bar{\theta}=2\pi$ are equivalent. Figure 1 shows a single trajectory, obtained by numerical integration of (1) from a random initial condition. The trajectory lies on a torus, defined implicitly by $I = S_{12}S_{23}S_{31} = $const. The motion is quasiperiodic, with a rapid advance along the $\bar{\theta}$ direction and a slow winding around the torus. Other trajectories lie on tori nested either inside or outside of that shown in Fig. 1. Thus the whole phase space is filled with invariant tori. The limiting torus at the center is a periodic orbit, corresponding to the "splay-phase" or "antiphase" solution [1–8,11].

Now we consider (2) as $N \rightarrow \infty$. By analogy with other integrable systems, one expects that the continuum limit should support solitary waves. The appropriate limiting system is an evolution equation for $\rho(\theta,t)$, the number density of oscillators, viewed here as particles moving around the unit circle. Conservation of oscillators yields $\partial\rho/\partial t = -\partial(\rho v)/\partial\theta$, where the velocity field $v(\theta,t)$ is given self-consistently by $v(\theta,t) = \int_0^{2\pi}\cos(\varphi - \theta)\rho(\varphi, t)d\varphi$, from the infinite-$N$ limit of (2). If we seek traveling waves of the form $\rho(\theta - Ut)$, we find a one-parameter family of solutions $\rho(\theta - Ut) = (2\pi)^{-1}V/[U - A \times \cos(\theta - Ut)]$ and $v(\theta - Ut) = A\cos(\theta - Ut)$, where $U = \frac{1}{2}(1 + A^2)$ and $V = \frac{1}{2}(1 - A^2)$. Here A measures the amplitude of both density and velocity modulations. When $A=0$, the oscillators are uniformly distributed on the circle and are motionless (recall that we are in a rotating frame); this is the splay-phase solution. For $0 < A < 1$, the density is peaked about the point $\theta = Ut$ and rotates around the circle at constant speed U, while the individual oscillators move nonuniformly, hesitating and accelerating according to the modulations in $v(\theta - Ut)$. Finally, as $A \rightarrow 1$, the density approaches a rotating δ function, corresponding to the periodic in-phase state.
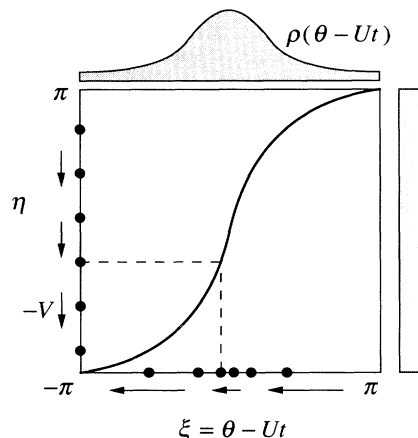

$\xi = \theta - Ut$

FIG. 2. In a frame rotating with the solitary wave $\rho(\theta - Ut)$, the individual oscillators (dots) move backward nonuniformly (horizontal arrows). The motion is slow where the density is high. A nonlinear change of coordinates (see text) converts the nonuniform motion to rigid rotation at speed $-V$ (vertical arrows), and the oscillators become evenly spaced. The shaded regions show the density before and after the transformation.

Now we introduce a change of coordinates that will play a key role in the rest of the analysis. This transformation arises naturally here in the continuum limit, but its main application is to the finite-N system (2). Consider a coordinate system $\xi = \theta - Ut$ rotating with the wave. The individual oscillators then have relative velocity $\dot{\xi} = v(\xi) - U < 0$. Suppose that we wish to convert this nonuniform motion into *uniform* rotation with respect to a new angular variable η. This can be achieved [20] by the transformation $\tan(\xi/2) = \beta\tan(\eta/2)$, where $\beta = (1 + A)/(1 - A)$ and $\dot{\eta} = -V$, with V defined as above. The oscillators become evenly spaced in the new coordinates and the density becomes uniform (Fig. 2). Now we freeze the oscillators by introducing another rotating coordinate $\psi = \eta + Vt$. Thus the overall transformation is $\tan[(\theta - Ut)/2] = \beta\tan[(\psi - Vt)/2]$. The claim is that if the oscillators have speed $v(\theta - Ut)$ and density $\rho(\theta - Ut)$ in the θ coordinate system, they will appear frozen and uniformly distributed in ψ.

This transformation can be generalized to simplify *all* the solutions of the continuum system, not just the solitary waves [20]. More importantly, it can be extended to the finite-N system (2). Given any solution $\{\theta_i(t)\}$ of (2), let

$$\tan\frac{\theta_i(t) - \Theta(t)}{2} = \beta(t)\tan\frac{\psi_i - \Psi(t)}{2}, \qquad (3)$$

for $i = 1, \ldots, N$. The evolution of $\Theta(t)$, $\Psi(t)$, and $\beta(t)$ is unknown for now, but will be determined later as part of the solution. The angles ψ_i are assumed to be frozen, by analogy with the earlier results. The transformation (3) should seem optimistic: It implies that an arbitrary solution of (2), which consists of N time-dependent functions

$\theta_i(t)$, can be generated by just *three* functions $\Theta(t)$, $\Psi(t)$, and $\beta(t)$, and N constants ψ_i. We now verify that this remarkable reduction is possible.

The first step is to show that (2) is satisfied identically, if Θ, Ψ, and β are chosen appropriately. After solving (3) for $\theta_i(t)$ and substituting in (2), we obtain $c_1(t) + c_2(t)\cos[\psi_i - \Psi(t)] + c_3(t)\sin[\psi_i - \Psi(t)] = 0$, where the coefficients $c_k(t)$ are complicated but *independent* of $i = 1, \ldots, N$. Hence the equations above are satisfied *identically* for all i and all t if $c_k(t) \equiv 0$, for $k = 1, 2, 3$. These three conditions imply [20]

$$\dot{\gamma} = \frac{(1-\gamma^2)^{3/2}}{N} \sum_j \frac{\sin(\psi_j - \Psi)}{1 - \gamma\cos(\psi_j - \Psi)}, \tag{4a}$$

$$\gamma\dot{\Psi} = \frac{(1-\gamma^2)^{1/2}}{N} \sum_j \frac{\gamma - \cos(\psi_j - \Psi)}{1 - \gamma\cos(\psi_j - \Psi)}, \tag{4b}$$

$$\gamma\dot{\Theta} = \frac{1}{N} \sum_j \frac{\gamma - \cos(\psi_j - \Psi)}{1 - \gamma\cos(\psi_j - \Psi)}, \tag{4c}$$

where the new variable $\gamma = (\beta^2 - 1)/(\beta^2 + 1)$ has been introduced for convenience. Thus, if $\Theta(t)$, $\Psi(t)$, and $\gamma(t)$ evolve according to (4), then the $\{\theta_i(t)\}$ defined by (3) are guaranteed to solve (2) for all t.

Equation (4) also suggests natural constraints on the constants $\{\psi_i\}$. If $\gamma = 0$, then (4b) and (4c) are satisfied identically in Ψ if and only if $\sum_j \cos\psi_j = \sum_j \sin\psi_j = 0$. From now on, assume the $\{\psi_i\}$ satisfy these constraints.

To finish the reduction, we show that *all* solutions of (2) can be generated by (3), subject to the constraints on $\{\psi_i\}$. It suffices to check that for a given set of initial conditions $\{\theta_i(0)\}$, we can always find corresponding constants $\{\psi_i\}$ and initial conditions $\Theta(0)$, $\Psi(0)$, and $\gamma(0)$ for (4). After solving (3) for ψ_i and imposing the constraints, we find that $\Theta(0)$ and $\gamma(0)$ must satisfy

$$\sum_j [\gamma + \cos(\theta_j - \Theta)]/[1 + \gamma\cos(\theta_j - \Theta)] = 0,$$

$$\sum_j [\sin(\theta_j - \Theta)]/[1 + \gamma\cos(\theta_j - \Theta)] = 0$$

simultaneously. A topological argument [20] based on index theory proves that these equations have a solution in the allowed region $0 \le \gamma(0) < 1$, $0 \le \Theta(0) < 2\pi$, for all generic $\{\theta_i(0)\}$ [23]. Hence $\Theta(0)$ and $\gamma(0)$ can be found. On the other hand, $\Psi(0)$ is arbitrary. To fix it, we must impose a third constraint on the $\{\psi_i\}$, e.g., $\sum_j \psi_j = 0$. Then (3) shows that $\Psi(0)$ and $\{\psi_i\}$ are determined by $\Theta(0)$, $\gamma(0)$, and $\{\theta_i(0)\}$, as required.

Thus the integration of the N-dimensional system (2) has been reduced to the integration of the three-dimensional system (4). More precisely, we have shown that (2) is equivalent to an $(N-3)$-parameter family of three-dimensional systems. The $N - 3$ parameters are the $\{\psi_i\}$, subject to the three constraints imposed above.

The final part of the argument is a proof that the solutions of (4) are confined to 2-tori. First, Eqs. (4a) and (4b) show that γ and Ψ are decoupled from Θ; in fact,

the (γ, Ψ) subsystem is equivalent to a Hamiltonian system with 1 degree of freedom. To see this, let $\alpha = (1 - \gamma^2)^{-1/2}$. Then $\dot{\alpha} = -\partial H/\partial \Psi$, $\dot{\Psi} = \partial H/\partial \alpha$ where

$$H(\alpha, \Psi) = \frac{1}{N} \sum_{j=1}^N \ln[\alpha - (\alpha^2 - 1)^{1/2}\cos(\psi_j - \Psi)]. \tag{5}$$

The appropriate phase space for (5) is a cylinder because H is 2π periodic in Ψ. For this Hamiltonian flow on a cylinder, the trajectories are closed orbits (or fixed points, in exceptional cases). Hence $\alpha(t)$, and therefore $\gamma(t)$, is periodic in t. Then (3) implies that the phase differences $\theta_i(t) - \theta_j(t)$ are periodic as well.

Second, consider the dynamics of $\Theta(t)$. Suppose that after one period T of $\gamma(t)$ and $\Psi(t)$, Θ shifts by some amount, say $\Theta(t + T) = \Theta(t) + 2\pi W$. Since the right-hand side of (4) is independent of Θ, the same shift occurs on the next cycle, and so on. Thus $\Theta(t + kT) = \Theta(t) + 2\pi k W$, for $k = 0, 1, 2, \ldots$. Therefore the solutions of (4) are typically quasiperiodic, with winding number W. This completes the proof that the trajectories of (4), and hence of (2), are typically confined to 2-tori.

Figure 3 illustrates these results for the case $N = 4$. As in Fig. 1, we have transformed coordinates to an orthogonal system based on the mean phase $\bar{\theta}$ and $N - 1$ relative coordinates (phase differences). Only the phase differences are shown in Fig. 3. All orbits are closed, as expected from our result that the phase differences are
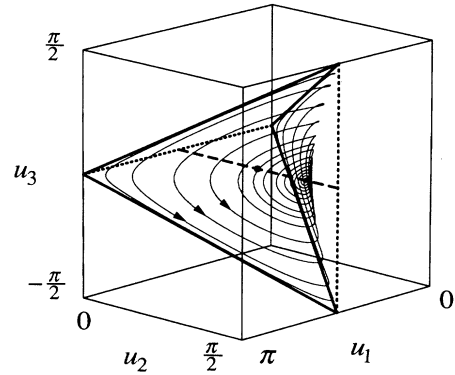

FIG. 3. Three-dimensional projection of the phase portrait for $N = 4$. The coordinates are $\bar{\theta} = \frac{1}{4}\sum_j \theta_j$, $u_1 = \frac{1}{4}(\theta_1 + \theta_2 - \theta_3 - \theta_4)$, $u_{2,3} = \frac{1}{4}(\pm \theta_1 \mp \theta_2 + \theta_3 - \theta_4)$. Only (u_1, u_2, u_3) are shown, since their dynamics are independent of $\bar{\theta}$. The canonical invariant region $\theta_4 \le \theta_3 \le \theta_2 \le \theta_1 \le \theta_4 + 2\pi$ defines a tetrahedron. Notation: dotted edges of the tetrahedron = two pairs of equal θ's; solid edges = three equal θ's; vertices = in-phase solutions; dashed bar inside = filament of "incoherent" states; dot at center = splay-phase state. Trajectories were generated by integrating (4) for several values of $\gamma(0)$, with $\Psi(0) = \Theta(0) = 0$, $\psi_{1,4} = \pm \frac{9}{10}\pi$, and $\psi_{2,3} = \pm \frac{1}{10}\pi$. Changing $\gamma(0)$ produces closed orbits of different amplitudes which together form a warped surface. $\gamma(0) = 0$ yields an incoherent state while $\gamma(0) \to 1$ produces a large orbit that passes by all four vertices, and hence is near the in-phase state.

periodic. (We do not see 2-tori, since we have projected out the $\bar{\theta}$ motion.) The orbits encircle the filament of "incoherent" fixed points, defined by the conditions $\sum_j \cos\theta_j = \sum_j \sin\theta_j = 0$. As $\gamma(0)$ increases, the orbits expand and sweep out a warped surface. A different choice of $\{\psi_i\}$ would produce a different surface; these surfaces are stacked side by side, skewered by the incoherent filament. On each surface the dynamics is Hamiltonian.

Our methods also yield the first global stability results for the more general system

$$\dot{\theta}_i = \frac{1}{N} \sum_{j=1}^{N} \cos(\theta_j - \theta_i - \delta), \quad i = 1, \ldots, N. \tag{6}$$

Equation (6) arises as the averaged system for both the complex Ginzburg-Landau equation with weak global coupling [15], and for series arrays of Josephson junctions weakly coupled through an RLC load [24]. The value of δ depends on the particular parameters in the physical system. Two cases are familiar: When $\delta = \pi/2$, Eq. (6) is the infinite-range XY model at zero temperature, and when $\delta = -\pi/2$, (6) is the antiferromagnetic XY model. But for other values of δ, we do not have gradient (relaxational) dynamics and standard techniques do not apply. Our analysis goes as before, starting from the transformation (3), and once again we find $N-3$ constants of motion, namely, the $\{\psi_i\}$ subject to their three constraints. The new feature is that H is no longer conserved; now it plays the role of a Liapunov function. We find $\dot{H} = P(\alpha, \Psi, \{\psi_i\}) \sin\delta$, where P is a positive definite function [20]. The trajectories are still confined to warped surfaces, but now they move in spirals rather than closed orbits. For $\sin\delta > 0$, H increases and almost all solutions spiral out to the in-phase state, while for $\sin\delta < 0$ they spiral down to an incoherent state predetermined by $\{\psi_i\}$. In the original nonrotating frame, these incoherent states would appear as an attractive manifold of marginally stable periodic states. Such states have been detected numerically [3,13].

After this paper was submitted, we noticed that the same analysis gives $N-3$ constants of motion for the *full equations* governing series arrays of overdamped, identical Josephson junctions (not only for the averaged counterparts studied here). This result [20] explains much of the neutral stability and phase space structure that has been detected previously, and it also provides bounds on the dynamical complexity of Josephson arrays. For instance, an array of N overdamped junctions with an RLC load has phase space dimension $N+2$; since there are $N-3$ constants of motion, the dynamics are effectively five dimensional. Hence any chaotic attractors must have a dimension ≤ 5, for all N. Unfortunately, our methods fail for underdamped junctions; this case is important for future study, as is that of nonidentical junctions.

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