Type Specification of Chaos

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Quasiregularity of Hamiltonian chaos is approached using the concept of the type of a chaotic orbit, i.e., the sequence of resonances visited by the orbit. A simple method is introduced for determining the type in the case of kicked-rotor Hamiltonians. First applications of this method are given in the calculation and study of quantities characterizing "vague tori" within resonances, namely, survival probabilities and average dwelling times. Results concerning a newly defined disorder parameter, characterizing quasiregularity, are briefly reported.

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Generic Hamiltonian systems exhibit regular and chaotic motions interleaved on all scales of phase space. The regular motion takes place on ordered and/or stable orbits (periodic orbits, KAM tori, cantori, and homoclinic orbits) associated, usually, to an infinite hierarchy of islands around islands [1]. This ordered structure seems to persist for arbitrarily strong chaos [2], and affects significantly the chaotic motion. A typical chaotic orbit looks like a random sequence of "quasiregular" segments, each resembling some ordered orbit in its immediate vicinity.

This key feature of Hamiltonian chaos has important manifestations, which have been the subject of several investigations during the last decade. Quasiregular segments of chaotic orbits have been associated with "vague tori" [3], characterized by "approximate constants of motion" in some time interval. Transitions from one "vague torus" to another, known as "tori jumps" [4], give rise to an intermittent-like behavior, with the laminar phases corresponding to the quasiregular segments. These segments may be identified by a spectral analysis [5] of the chaotic motion, each segment being associated with the fundamental frequency of some ordered orbit, in whose neighborhood the segment lies. The lengths of the segments may be arbitrarily long, leading to slow decays of correlations [6] and related phenomena.

In all these investigations, however, the notion of quasiregularity has been approached on a semiempirical basis. The results thus obtained are approximate and somehow vague. For a more complete and detailed analysis, accurate calculations, and to avoid ambiguities, a more systematic approach is required. This should be based on a precise definition of quasiregularity, namely, how exactly the quasiregular segments are associated with ordered orbits of well defined dynamical specifications. Clearly, one needs first a partition of phase space into regions containing, in a natural way, the ordered-motion components on all (existing) classes [1] of the island-around-island hierarchy. The sequence of quasiregular segments for a chaotic orbit can then be identified with the sequence of regions visited by the orbit. In order to apply this definition of quasiregularity to several problems of interest, e.g., those mentioned above, it is most important to devise effective methods for determining the sequence of regions visited by each given (numerical) chaotic orbit.

In this Letter we study a precise definition of quasiregularity for a typical model system by introducing a simple method for determining the sequence of quasiregular segments. The model system considered, exhibiting the generic mixture of regular and chaotic motions, is the family of area-preserving maps

$$p_{t+1}=p_t+Kf(x_t)\,,\qquad x_{t+1}=x_t+p_{t+1}\ (\mathrm{mod}\,1)\,, \tag{1}$$

corresponding to kicked-rotor Hamiltonians. Here K is the "kicking" parameter, and the impulse function $f(x)$ is periodic in x with period 1 ($2\pi x$ is the rotation angle). We assume $f(x)$ to be antisymmetric, so that the maps (1) are reversible [7]. The basic regions of ordered motion for (1) have been identified recently [8] with the *resonances* [9–12]. As defined in Ref. [9], a resonance (m,n), m and n being relatively prime integers, is a chain of n closed regions (the "islands") bounded by pieces of stable and unstable manifolds of a hyperbolic *ordered* [13] periodic orbit of winding number m/n. Out of the n islands, $n-1$ are the first $n-1$ map preimages of the nth island, to be referred to as the "central" island. For a large class of interesting functions $f(x)$ (to which we shall restrict our attention), it appears that the resonance boundaries (pieces of stable and unstable manifolds) can always be chosen [9] so that the central island is symmetrically positioned around the "dominant" symmetry line [7] $x=0$ of (1). Moreover, numerical [9] and analytical [12] evidence indicates that, with this choice, different resonances do not overlap. Now, in the absence of KAM tori, the total area of resonances, for all m/n, is equal to the area of phase space [10]. It then follows that the resonances give a complete partition of phase space, so that a general chaotic orbit must have all its points within resonances. The *type* [8] τ of the chaotic orbit is the sequence of resonances visited by it, and is denoted by $\tau=\ldots,(m_r,n_r)_{q_r},(m_{r+1},n_{r+1})_{q_{r+1}},\ldots$, where q_r is a positive integer denoting the number of consecutive visits of resonance r, (m_r,n_r) (or number of rotations performed in r). The type specification of chaos is a definition of quasiregularity on the lowest class (class 0) [1] of the island-around-island hierarchy. To extend this definition to higher classes, one has to partition each resonance into regions containing the island-around-island structure. However, while the resonance partition seems to be generic for the maps (1), this is not the case for the secondary partitions, which may not even exist for some maps (see Ref. [12]). The resonance partition turns out also to be sufficient for many purposes. We shall therefore restrict ourselves here to the definition of quasiregularity given by the type specification.

A trivial method for determining the type of a (numerical) chaotic orbit of (1) is to calculate first the resonance boundaries (pieces of stable and unstable manifolds) for all winding numbers m/n with, say $n \le n_0$, where n_0 is large enough, and to determine then in which resonance each point of the orbit lies. This method is clearly quite tedious, since it requires very accurate numerical calculations of the resonance boundaries as large sets of data points, which have to be stored for determining the type. Moreover, as the resonance boundaries vary with K, these calculations have to be repeated for each value of K considered. These difficulties can be avoided by using a much simpler method, based on the following observation: With the standard choice of the resonance boundaries (see above), the central island of each resonance is symmetrically positioned around the dominant symmetry line $x=0$. This line then divides the central island into two halves: the $\mathcal{R}$ half to the right of $x=0$ and the $\mathcal{L}$ half to the left of $x=1$. In addition to the type specification τ, one may also specify which of the regions $\mathcal{L}$ or $\mathcal{R}$ is visited, for each resonance in τ. This leads to a coding scheme $\{\tau,\mathcal{L}-\mathcal{R}\}$, which was considered recently [12] in a special case of τ for the sawtooth map, with $f(x) = x-0.5$. In general, we denote [12] by X_t the value of x_t lifted to the real axis, i.e., $X_t=x_t+a_t$, where $a_t=[X_t]$ is the greatest integer less than, or equal to, X_t. One then obtains from (1) the "Newton" equation

$$x_{t+1}-2x_t+x_{t-1}=Kf(x_t)-b_t\,, \tag{2}$$

where

$$b_t=a_{t+1}-2a_t+a_{t-1}\,.$$

Our main statement is now that the code $\{\tau,\mathcal{L}-\mathcal{R}\}$, in particular the type τ, is uniquely determined by the sequence a_t (or b_t, with a_0 and a_1 fixed by the initial conditions). The general stages of the proof are as follows (details will be given elsewhere [14]). First, a_t may be easily expressed in terms of the code $\{\tau,\mathcal{L}-\mathcal{R}\}$ (see a special case in Ref. [12]). This expression does not depend on the details of the map [K or $f(x)$], since it is just a consequence of the common symmetry and topology shared by the resonances in the maps (1). Next, suppose that two orbits, with different codes $\{\tau,\mathcal{L}-\mathcal{R}\}$, give the same a_t by this expression. Since the expression is map independent, this is also true for the sawtooth map, provided the two codes exist for this map. This is indeed the case, since it is easy to show that any allowed code $\{\tau,\mathcal{L}-\mathcal{R}\}$, for a general map (1), also exists for the sawtooth map if K is sufficiently large. But the sequence a_t is known [12] to be a symbolic dynamics for the sawtooth map; i.e., it determines the orbit completely. This leads to a contradiction, and our statement is thus proven. The type τ can then be easily determined as follows: (a) For each t, calculate from (2) the sequence $b_{t-N},\ldots,b_{t+N}$ ($2N+1$ elements) for N large enough (see below). (b) Calculate the corresponding orbit $x_t^{(s)}$ for a sawtooth map with a properly chosen parameter K_s, using the known [12] relation between $x_t^{(s)}$ and b_t:

$$x_t^{(s)} \approx \frac{1}{2}+\sum_{l=-N}^{N}\frac{\lambda_s^{-|l|}}{\lambda_s-\lambda_s^{-1}}b_{t+l}\,,\qquad \lambda_s=1+\frac{K_s+(K_s^2+4K_s)^{1/2}}{2}\,. \tag{3}$$

The parameter K_s is chosen so that the sequence b_t is allowed for this sawtooth map, i.e., that [12] $0\le x_t^{(s)}\le 1$ for all t. We notice from (2) that $|b_t|\le K\bar{f}+2$, where $\bar{f}$ is the maximal value of $f(x)$. Using then the exact formula for $x_t^{(s)}$ [Eq. (3) with $N=\infty$], it is easy to see that b_t is certainly allowed if one chooses

$$K_s=2[K\bar{f}+2]\,. \tag{4}$$

(c) Finally, the type τ is straightforwardly determined from the orbit (3) and the analytic expressions for the resonance boundaries of the sawtooth map [12]. Here all the resonances (m,n) with $n\le n_0$ are considered, and n_0 is chosen in consistency with the accuracy N of the approximation in (3).

As an example, let us use this method to determine the type of a long chaotic orbit in the standard map, with $f(x)=-\sin(2\pi x)/2\pi$. In this case, $K_s=4$ in (4) for all $K<2\pi$. This is a most interesting domain of parameters for the standard map. Now, the error in the approximation (3) is of the order of λ_s^{-N}, and the width [12] of the highest-order resonances considered is of the order of $\lambda_s^{-n_0}$. Thus, when computing in double-precision arithmetic with $K_s=4$, it is consistent to choose values of N and n_0 not larger than $N\approx n_0\approx 20$. The type is then determined up to resonance order n_0. Provided K is not too small, the probability that the orbit will visit resonances of order higher than n_0 is usually quite small. In fact, we find that this is the case even for values of K close to the critical value [15] $K_c\approx 0.9716\ldots$. An example is shown in Fig. 1 for $K=1.2$. The type of the orbit is plotted here, in a natural representation, up to the first crossing time of a golden-mean cantorus.

We now consider some applications of the method. As

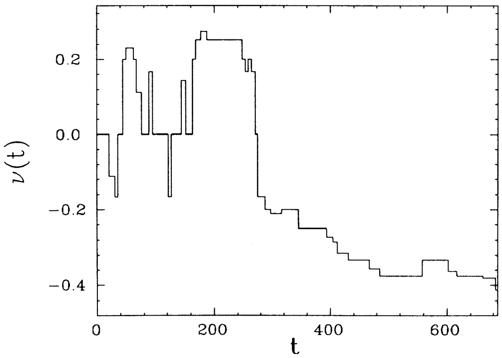

FIG. 1. Type of a chaotic orbit for the standard map at $K=1.2$, with initial condition $(x_0,p_0)=(0.5001,0.0)$. The type is represented here by the winding number $\nu=m/n$ of the resonance visited as a function of time t. This is a steplike function, assuming a constant value m_r/n_r in a time interval of length $q_r n_r$ [q_r rotations in resonance (m_r,n_r)]. This function is plotted up to the first crossing time of a golden-mean cantorus with $\nu=(\sqrt{5}-3)/2$.

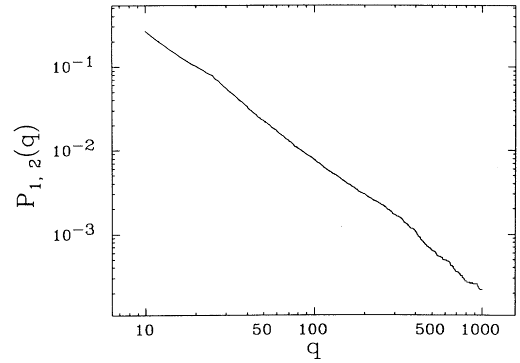

FIG. 2. Log-log plot of the survival probability in resonance (1,2) of the standard map at $K=1.5$, calculated by collecting chaotic-orbit segments trapped in (1,2). These segments are identified using our method for determining the type, as described in the text. The ensemble of chaotic orbits considered has initial conditions on a 30×30 grid localized around the hyperbolic fixed point (0.5,0.0). The length of each orbit is 25000 iterations.

a first application, we study the survival probability [6] in a resonance (m,n), to be denoted by $P_{m,n}(q)$. This is the probability that a chaotic orbit, entering (m,n) at time $t=0$, will remain in (m,n) at least until $t=qn-1$; i.e., it will perform at least q rotations. $P_{m,n}(q)$ is a good characterization of the "vague torus" [3] associated with resonance (m,n). The slower $P_{m,n}(q)$ decays with q, the less vague the "vague torus" is. There is also a simple relation [6] between $P_{m,n}(q)$ and a local correlation function $C_{m,n}(q)$ in resonance (m,n). $P_{m,n}(q)$ is easily calculated from the trapping statistics [6], i.e., the probability that an orbit segment trapped in (m,n) has length qn. The latter probability can be accurately determined by following a large ensemble of sufficiently long chaotic orbits, using our method. As an example, we plot in Fig. 2 $P_{1,2}(q)$ for the standard map with $K=1.5$. $P_{1,2}(q)$ appears to decay algebraically as $q^{-\alpha}$, with $\alpha\approx1.6$. On the other hand, for $K=3$ (see Fig. 3), the initial decay of $P_{1,2}(q)$ is clearly exponential, as $\exp(-\beta q)$, with $\beta\approx1$. An initial exponential decay of $P_{m,n}(q)$ has been observed to emerge, for all the resonances considered, when K is increased beyond the accumulation point $K_{m,n}$ of the period-doubling sequence of the central elliptic point in the resonance. As K approaches $K_{m,n}$ from below (above), α (β) increases (decreases). The occurrence of an initial exponential decay of $P_{m,n}(q)$, following the destruction of the main (frequency $\frac{1}{2}$) island-around-island hierarchy [1], is physically obvious. In fact, in this case, the resonance interior appears to be completely chaotic (without stability regions) for short time scales. However, for sufficiently long time scales, the stickiness effect [6] near the boundaries of small, secondary stability regions in (m,n) should manifest itself in an algebraic [16] asymptotic decay of $P_{m,n}(q)$. The presence of such an asymptotic decay could not be ascertained because of the limited accuracy of our calculations. We hope to report the results of more accurate calculations in a future work.

As a second application, we consider the average dwelling time $T_{m,n}$ in a resonance (m,n), which we define as the normalized quantity $T_{m,n}=M_{m,n}/M$. Here $M_{m,n}$ is the number of points belonging to trapped-orbit segments in (m,n), and M is the total number of orbit points in a large ensemble of sufficiently long chaotic orbits (see caption of Fig. 2). Clearly, $T_{m,n}$ is the average fraction of time spent in the vague torus (m,n), performing there quasiregular motion with winding number m/n. Our calculations for the standard map, in quadruple-precision arithmetic, indicate that $T_{m,n}$ depends on (m,n) and K only through the residue [15] R of the hyperbolic ordered

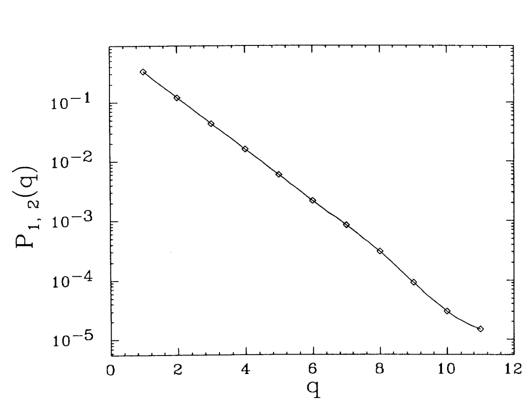

FIG. 3. Semilogarithmic plot of the survival probability in resonance (1,2) of the standard map at $K=3.0$, calculated as described in the caption of Fig. 2. The length of each chaotic orbit considered is here 20000 iterations.

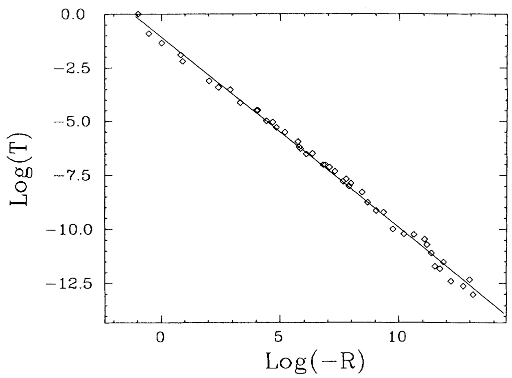

FIG. 4. Log-log plot of the average dwelling time in resonances of the standard map at $K=1.5$, as a function of the residue R of the corresponding hyperbolic ordered periodic orbits. All resonances (m,n) with $n \leq 26$ and $m \leq n$ have been considered. The straight line best fitting the data has slope ≈ -0.89. Approximately the same slope has been measured also for other values of K.

periodic orbit. The function $T(R)$ appears to satisfy a scaling law $T(R) \propto |R|^{-\gamma}$, with $\gamma=0.87 \pm 0.02$. An example is shown in Fig. 4 for $K=1.5$. Now, it is known [9] that the area A of a resonance also scales with R as $A(R) \propto |R|^{-0.94}$. This is evidence for ergodicity in the chaotic region (equal times spent in equal areas), provided the fraction of resonance area occupied by the chaotic component scales approximately as $|R|^{0.07}$. This is plausible, but, of course, remains to be checked independently.

In conclusion, we have presented a simple method for determining the type of chaotic orbits in reversible area-preserving maps. As first applications of this method, we have studied the survival probabilities and average dwelling times within resonances, as characterizations of "vague tori." Work is in progress concerning two other interesting applications of the method: (a) spectral analysis of chaotic orbits [5] trapped within resonances, so as to probe the island-around-island structure; and (b) studying the behavior of a disorder parameter, characterizing quasiregularity, as a function of dynamical quantities of interest. Such a parameter may be associated with unstable periodic orbits (UPOs) in the chaotic region, and defined at fixed winding number ν. Let the type of a UPO be [8] $\tau=[(m_1,n_1)_{q_1},\ldots,(m_R,n_R)_{q_R}]$ (R resonances involved) with winding number $\nu=\sum_{r=1}^{R} q_r m_r / \sum_{r=1}^{R} q_r n_r$. We then define the disorder parameter $\Delta\nu$ as follows:

$$\Delta\nu = \left[\sum_{r=1}^{R} q_r n_r (m_r/n_r - \nu)^2 \Big/ \sum_{r=1}^{R} q_r n_r \right]^{1/2} .$$

We have performed [14] extensive calculations of $\Delta\nu$ for the tent map [17], using our method to determine the types of UPOs. At a fixed value of ν, the function $\Delta\nu(S)$, where S is the action of a UPO per iteration, appears to be much more well behaved than $\sigma(S)$, the Lyapunov exponent. In particular, near $\Delta\nu=0$, $\Delta\nu(S)$ is always monotonously increasing. This is consistent with the known [7] fact that the hyperbolic ordered periodic orbits (with, of course, $\Delta\nu=0$) minimize S at fixed ν. We expect these results to extend to more generic maps, like the standard map. The parameter $\Delta\nu$ may turn out to be useful in several interesting applications, e.g., in a systematic resummation of Gutzwiller trace formula for semiclassical quantization [18].

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