Truncated Icosahedral Gravitational Wave Antenna

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We propose a new type of resonant-mass gravitational wave detector, a truncated icosahedral gravitational wave antenna. It will be omnidirectional, and able to measure the direction and polarization of a detected wave. We solve a model for this system, calculate the strain noise spectrum, and conclude that its angle-averaged energy sensitivity will be 56 times better than the equivalent bar-type antenna with the same noise temperature.

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Confirmed detection of gravitational waves from astrophysical sources will found a new astronomy, and allow direct investigation of the gravitational force under extreme conditions. The best current antennas, such as the LSU detector [1], are sensitive enough to detect a gravitational collapse in our galaxy, but the conventional wisdom is that we need to look at least 10^3 farther in distance to have an "assured" event rate of several per year. This requires reducing the energy resolution by 10^6. The best known methods for improving cryogenic resonant-mass detectors will contribute by reducing the energy resolution in proportion to the reduction of the noise temperature T_n from its current value of ~ 7 mK. However, it is commonly believed that quantum noise will present a formidable barrier for improvement by more than 10^5, not quite enough for assured detection.

However, there are other ways to improve resonant-mass antennas that are independent of T_n. One way is to increase the cross section. Another is to make multiple antennas, aimed in different directions, so every source direction and polarization will be within at least one antenna pattern. This method adds the ability to determine source direction and polarization. *A spherical antenna promises to provide all three improvements in a single instrument.*

The question becomes the actual magnitude of these improvements. We have invented a design for a nearly spherical antenna, which we call a truncated icosahedral gravitational wave antenna (TIGA), that provides an elegant solution to certain complications of a spherical antenna, and therefore lets us calculate the quantitative improvement. We conclude that a TIGA will be about 56 times more sensitive in energy than the equivalent bar-type antenna with the same noise temperature T_n. Combined with a quantum limited T_n, this is a sufficient factor to increase our range by more than the desired factor of 10^3. If we assume construction of a *set* of detectors for different frequencies, or "xylophone," the sensitivity is further improved and wave form information can be obtained.

It was recognized long ago [2] that a sphere is a very natural shape for a resonant detector of gravitational waves. A free sphere has 5 degenerate quadrupole modes of vibration that will interact strongly with a wave, a bar has only 1. Each free mode can act as a separate antenna, oriented towards a different polarization or direction. Wagoner and Paik [3] found a set of equations to determine the source direction from the free mode amplitudes. They also calculated the angle-averaged energy absorption cross section of a sphere. Compared to a bar with the same quadrupole mode frequency and a typical length/diameter$=4.2$, the improvement in cross section is about a factor of 60.

This result has been ignored, perhaps because a free spherical resonator is not a practical detector. The first requirement for practicality is a set of *secondary* modes or mechanical resonators. Every successful cryogenic bar-type detector has required one; it acts as a mechanical-impedance transformer between the primary mode of the antenna and the actual motion sensor, supplying an essential increase in the coupling. We expect that a sphere with 5 primary modes will require at least 5 secondary resonators. The second requirement is a clear method for orientational deconvolution of the signal, so we can determine its direction and polarization. The third requirement is a way to quantify the noise from multiple motion sensors.

It is not hard to imagine that the presumed advantages of a sphere might be lost due to these complications. To convince the skeptics, including ourselves, a detailed proposal and calculation is necessary. Our analysis is an extension, to multimode antennas, of the type developed by Michelson and Taber [4,5].

To derive a first order theory for a practical detector, we start with the elastic theory for a free sphere [6] with mass m_s and radius R, and consider only the 5 lowest order (degenerate) quadrupole modes, with resonant frequency ω_s. It is convenient to define an amplitude vector $\mathbf{a}(t)$, whose components are the amplitudes of these modes.

The successful secondary resonators used on bar antennas couple to motion *normal* to the antenna surface, so we restrict our consideration to that type. The one-dimensional resonators are assumed identical, with mass m_t and spring constant k_t, and tuned to the sphere frequency, so that $k_t/m_t=\omega_s^2$. We define a transducer vector $\mathbf{q}(t)$, whose components are the radial displacement of each resonator mass, relative to the sphere surface. Because the radial motion of the mth free sphere mode is proportional to the spherical harmonic Y_{2m}, the geometric properties of a particular transducer arrangement are summarized by the $5\times J$ "pattern matrix" $\mathbf{B}$, defined by $B_{mj}\equiv Y_{2m}(\theta_j,\phi_j)$, where J is the number of resonators and (θ_j,ϕ_j) is the location of transducer j.

After some algebra, the model becomes a set of $5+J$ coupled harmonic oscillators, driven by the effective forces $\mathbf{F}$ applied to the free sphere modes, and resonator forces $\mathbf{f}$, applied between the resonator and the sphere surface.

$$\begin{bmatrix} m_s\mathbf{I} & \mathbf{0} \\ m_t\alpha\mathbf{B}^T & m_t\mathbf{I}\end{bmatrix}\begin{bmatrix}\ddot{\mathbf{a}}(t)\\ \ddot{\mathbf{q}}(t)\end{bmatrix}+\begin{bmatrix} m_s\omega_s^2\mathbf{I} & -k_t\alpha\mathbf{B} \\ \mathbf{0} & k_t\mathbf{I}\end{bmatrix}\begin{bmatrix}\mathbf{a}(t)\\ \mathbf{q}(t)\end{bmatrix} =\begin{bmatrix}\mathbf{I} & -\alpha\mathbf{B}\\ \mathbf{0} & \mathbf{I}\end{bmatrix}\begin{bmatrix}\mathbf{F}(t)\\ \mathbf{f}(t)\end{bmatrix}. \quad (1)$$

The constant α is determined by the elastic properties and the eigenfunctions.

The final readout of the system is a set of motion sensors, consisting of a transducer and a linear amplifier, one for each secondary resonator, which provide continuous outputs proportional to the displacements $\mathbf{q}(t)$.

For a gravitational wave traveling on the z axis, with the two polarization components $h_+(t)$ and $h_x(t)$, we calculate the effective force components to be

$$F_1=m_s(0.601R)\tfrac{1}{2}\ddot{h}_+(t),\quad F_2=m_s(0.601R)\tfrac{1}{2}\ddot{h}_x(t)\,,$$
$$F_3=F_4=F_5=0\,.$$

For a wave from an arbitrary direction, the rotation matrix needed to transform $\mathbf{F}$ into this form [3] provides *the complete orientational deconvolution* of the signal.

Thus a primary task of antenna design is to find the optimum way to infer the components of $\mathbf{F}(t)$ from noisy measurements of $\mathbf{q}(t)$, in presence of noise forces $\mathbf{f}(t)$. The equations can be solved for any arbitrary pattern $\mathbf{B}$ of transducers, but the result is so complicated, with all the components of $\mathbf{F}$ appearing in all the outputs $\mathbf{q}$, each with complicated frequency dependence, that deconvolution in both space and time looks to be very tedious, and optimization even more difficult.

Therefore we propose a resonator pattern with special symmetry: Six resonators arranged to match half the face-centers of a dodecahedron concentric to the spherical antenna. We also propose to replace the sphere mass by a *truncated icosahedron (TI)*, because it has the same symmetry as the dodecahedral array, plus it has extra faces suitable for mounting calibrators, suspension elements, etc. This shape is shown in Fig. 1 with the proposed resonator locations. It is interesting that this shape, arrived at because of its symmetry properties relative to the spherical harmonics of order 2, is also the shape of the C_{60} atom, or "buckyball." The special symmetry of the TI arrangement is signaled by the special property of its pattern matrix:

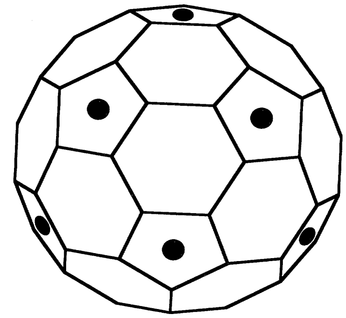

FIG. 1. The truncated icosahedral gravitational wave antenna (TIGA) with secondary resonator locations indicated.

$$\mathbf{B}\cdot\mathbf{B}^T=\frac{3}{2\pi}\mathbf{I}\,. \quad (2)$$

Unique among the patterns we have examined, the TI arrangement has a very simple spectrum of coupled-mode eigenfrequencies: there are two degenerate quintuplets, one downshifted to ω_-, and the other upshifted a nearly equal amount to ω_+, and a singlet remains at ω_s. The algebra to diagonalize Eq. (1) is lengthy but becomes straightforward after one finds a special choice for the eigenvectors of the degenerate modes and makes repeated use of Eq. (2).

That solution becomes even simpler if we define 5 linear combinations of the outputs, which we call the "mode channels," by

$$\mathbf{g}(t)\equiv\mathbf{B}\cdot\mathbf{q}(t)\,.$$

Then, in the frequency domain, the mode channels' responses to the forces are given by the remarkably simple expression

$$\mathbf{g}(\omega)=\left[\frac{c_+}{(\omega_+^2-\omega^2)}-\frac{c_-}{(\omega_-^2-\omega^2)}\right]\frac{F(\omega)}{(m_sm_t)^{1/2}} +\left[\frac{d_+}{(\omega_+^2-\omega^2)}-\frac{d_-}{(\omega_-^2-\omega^2)}\right]\frac{\mathbf{B}\cdot\mathbf{f}(\omega)}{m_t}\,,$$

where $c_\pm$ and $d_\pm$ are constants of order one that depend weakly on m_t/m_s.

Because $\mathbf{g}$ and $\mathbf{F}$ are proportional, *each mode channel is a direct readout for the corresponding component of the gravitational force* $\mathbf{F}$. Further, the frequency dependence is exactly like that of a bar-type antenna with one secondary resonator, so we can adopt familiar methods for filtering and optimization. The correlations of the force and additive noises are easily taken into account. Thus all the readout complications mentioned above are solved.

It might be argued that our result is questionable because the free modes cannot be exactly degenerate, the transducers cannot be exactly matched, etc. But when these effects are included, we expect to find that the overall sensitivity depends on them only in second order, for the same reasons that mistuning of similar parameters has proven to have only second order effects on the sensitivity of a bar with one resonator [7].

We find this configuration gratifying in its symmetry and simplicity, but there may be alternatives. The Stanford group is analyzing another configuration [8].

A noise model is needed to predict the signal-to-noise ratio. In this paper we consider only motion sensor noise. (The extension to thermal noise is straightforward.) We model the sensor noise in a generic fashion, similar to Price [9]. Each of the motion sensors is assumed to have an additive noise at its output whose spectral density (double-sided) is S_q. Each also applies a "back action" noise force to the resonator mass with spectral density S_f. We assume the sensors to be uncorrelated but identical. Then the sensor noise temperature T_n can be shown to be given by $k_b T_n \equiv (S_f \omega_s^2 S_q)^{1/2}$. We refer to it by the noise number $N \equiv k_b T_n / \hbar \omega_s$, the noise temperature normalized by the quantum of energy. The results also depend on the coupling parameter $r_n \equiv (S_f / \omega_s^2 S_q)^{1/2}$, called the noise resistance.

To display the results, we introduce the use of a strain noise spectrum $\tilde{h}(f)$ for the description of a resonant-type antennae, in close analogy to its use with the Laser Interferometer Gravitational Observatory (LIGO) prototypes [10]. It is the square root of the fictitious strain noise spectral density (single-sided) needed to mimic the observed noise present at *a single channel.* It has the virtue of allowing definite predictions for the detectability of an arbitrary signal wave form, assuming the noise is stationary.

The calculated strain noise is found to be proportional to $\sqrt{N}$, so that

$$\tilde{h}(f,N) = \sqrt{N}\tilde{h}(f,N=1)\,.$$

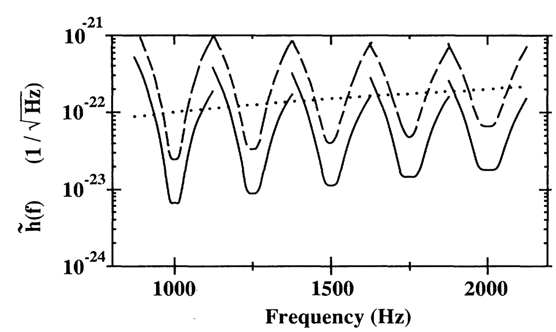

FIG. 2. The calculated strain noise spectrum $\tilde{h}(f)$ for various detectors. Solid lines: for a "xylophone" of TIGA detectors with quantum limited sensor noise, for a *single* channel (i.e., a single linear polarization arriving from an arbitrary direction). Dashed lines: a xylophone of equivalent bar antennas with quantum limited sensor noise, for the *optimum* orientation of the wave. Dotted line: for the first generation LIGO detector, for the *optimum* orientation of the wave [10].

The solid lines in Fig. 2 show $\tilde{h}(f,N=1)$, the strain noise for the quantum noise limited case, for a xylophone of TIGAs made of aluminum. The shape of the curves is determined by m_t/m_s and r_n, which were adjusted to give a consistent fractional bandwidth and a maximally flat curve. Parameters of the xylophone are shown in Table I.

For comparison, the corresponding results for the equivalent bars in the *most favorable* orientation for that same strain component are shown as the dashed lines in Fig. 2. The strain noise $\tilde{h}(f,N=1)$ for the equivalent bar is bigger by a factor of 3.9. We conclude that under equivalent conditions (i.e., equal noise numbers) a single channel of a TIGA will have $3.9^2 = 15$ times better energy resolution than the optimally oriented equivalent bar. This is nearly the same improvement calculated by scaling up the mass of the bar by this amount, so we conclude that a single channel of the TIGA suffers no signal-to-noise penalty due to the various complications in the readout.

The comparisons in Fig. 2 *understate* the overall advantage of the TIGA for the detection of gravitational waves. It has four more output channels that are optimally oriented for other polarizations and directions. For a bar detector, it is well known that averaging over source direction and polarization [11] leads to a loss of energy resolution, compared to the optimum, by a factor of $15/4 = 3.7$. Thus the net result is that the angle-averaged energy resolution of the TIGA is $3.7 \times 15 = 56$ times better than the equivalent bar detector (or about 7.5 times better in h).

The technical feasibility of approaching the quantum limited noise temperature, i.e., of reaching $N \approx 1$, is largely independent of whether the detector is a bar or a sphere. Evaluating the feasibility is not the subject of this paper, but we maintain that there have been experimental demonstrations of nearly all of the individual factors needed to reach the quantum limit, the latest being the demonstration, by the Rome group, of the cooling of a 2 tonne bar to 60 mK [12]. The challenge is the integration of these factors into a complete system.

By inspection, the xylophone also has the ability to pro-

TABLE I. Parameters for the "xylophone" of TIGA detectors shown as the solid lines in Fig. 1. The material is aluminum.

Frequency (Hz)	Radius (m)	TIGA mass (kg)	Resonator mass (kg)
1000	1.30	25 100	9.02
1250	1.04	12 800	4.62
1500	0.87	7400	2.67
1750	0.74	4700	1.68
2000	0.65	3100	1.13

vide substantial spectral information about the detected wave. Coherent recording of the outputs will allow relative phase measurement, hence reconstruction of the time dependence of the wave form.

Also shown for comparison in Fig. 2 (dotted line) is the predicted strain noise for the first generation LIGO detector [10] in its most favorable orientation for the same signal. It is evident that a xylophone of quantum limited TIGAs is significantly more sensitive over most of this frequency range, even without considering the extra information available about orientation. However, the predicted LIGO strain noise continues dropping, proportional to frequency f, down to $f \sim 100$ Hz. We conclude that the two detector types are complimentary, each having a frequency domain where the predicted sensitivities are superior.

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