Lattice Integrable Systems of Haldane-Shastry Type

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We present a new lattice integrable system in one dimension of the Haldane-Shastry type. It consists of spins positioned at the static equilibrium positions of particles in a corresponding classical Calogero system and interacting through an exchange term with strength inversely proportional to the square of their distance. We achieve this by viewing the Haldane-Shastry system as a high-interaction limit of the Sutherland system of particles with internal degrees of freedom and identifying the same limit in a corresponding Calogero system. The commuting integrals of motion of this system are found using the exchange operator formalism.

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Recently, the interest in spin systems of the Haldane-Shastry type as well as in integrable systems of particles with internal degrees of freedom has been revived [1–8]. The Haldane-Shastry model for spin chains and its SU(N) generalization consists of spins or, in general, SU(N) color degrees of freedom equally spaced around the unit circle with the Hamiltonian [1]

$$H=\sum_{i<j}\frac{1}{\sin^2[(x_i-x_j)/2]}P_{ij}\,, \tag{1}$$

where x_i are the positions of the spins and P_{ij} is the operator which exchanges the spins or colors of sites i and j. Haldane and Shastry found the antiferromagnetic ground state wave function of the system, which is similar in form to the ground state wave function of the Sutherland system of particles on the circle [9], as well as all energy levels for the system.

Although the above system was suspected to be integrable, and particular commuting integrals of the motion were sporadically found [2,10], a complete proof was lacking. Recently, however, Fowler and Minahan [11] showed the integrability of the system and derived the conserved quantities using a recently developed exchange operator formalism [12]. Their approach consists of working initially with a system of N bosons with internal degrees of freedom and no kinematics which sit on the N lattice sites and only allowing states with exactly one particle per site. Then every operator which is invariant under particle permutation must involve degrees of freedom on all lattice sites and can thus be substituted with a corresponding lattice operator. Integrability of this particle system, then, translates into integrability for the lattice system. These authors, then, consider the operators

$$\pi_i=\sum_{j\neq i}\frac{z_j}{z_{ij}}M_{ij}\,, \tag{2}$$

where the indices i, j now refer to particles,

$$z_i=\exp(2\pi i x_i)$$

and

$$z_{ij}=z_i-z_j\,.$$

M_{ij} are the operators which exchange the *positions* of particles, satisfying

$$M_{ij}x_i=x_jM_{ij},\quad M_{ij}x_k=x_kM_{ij}\quad(\text{for } i\neq k\neq j) \tag{3}$$

as well as the standard permutation group commutation relations among themselves. The Hamiltonian of the system is taken to be

$$H=\sum_{i<j}\frac{1}{\sin^2[(x_i-x_j)/2]}M_{ij}\,. \tag{4}$$

Using the commutation properties of M_{ij} and z_i one can show that the quantities

$$I_n=\sum_i \pi_i^n \tag{5}$$

commute among themselves and, if the lattice sites are equidistant, they also commute with the Hamiltonian. Therefore this system is integrable. Each operator M_{ij}, now, acting on a bosonic state translates into a spin exchange operator σ_{ij} for the particles [8]. Since the I_n are symmetric under particle permutation, every *particle* spin exchange operator they contain will translate into a *site* spin exchange operator P_{ij} and will reduce to the commuting conserved quantities of the corresponding Haldane-Shastry lattice system.

The above operators π_i considered by Fowler and Minahan are, in fact, identical in form to the corresponding operators considered by this author in the exchange operator formalism of the Sutherland problem [12], only lacking an explicit kinetic term. These operators are

$$\begin{aligned}\bar{\pi}_i&=p_i+il\sum_{j\neq i}\cot\left(\frac{x_i-x_j}{2}\right)M_{ij}+l\sum_{j\neq i}M_{ij}\\&=p_i-2l\pi_i\end{aligned} \tag{6}$$

(from now on we use barred symbols to represent quantities for the fully dynamical system of particles) while the Hamiltonian is

$$\bar{H}=\sum_i \tfrac{1}{2} p_i^2+\sum_{i<j} \frac{l(l-M_{ij})}{\sin^2(x_i-x_j)} . \quad (7)$$

If we rescale $\bar{\pi}_i$ by a factor of $-\frac{1}{2}l^{-1}$ and take the limit $l\to\infty$ we see that the kinetic term drops and we recover π_i. On the other hand, the leading term in H in this limit (of order l^2) becomes a nondynamical constant and can be subtracted away. The highest nontrivial term then becomes exactly of the form (4).

We see therefore that the lattice system can be viewed as the high-interaction limit of a corresponding Sutherland model. (The strength of the two-body interaction is of order l^2.) We must be careful, though. In fact, the limit $l\to\infty$ is the same as the *classical limit* $\hbar\to 0$ (this can be seen by restoring $\hbar$ into the problem) and, of course, the momentum does not become irrelevant in the classical limit. The point is that p_i as an operator has an unbound spectrum and therefore cannot be neglected no matter how large l becomes. In order to consistently drop it we must restrict our attention to states with no momentum excitations. The internal degrees of freedom then remain the only dynamical variables of the problem. This also means that in the classical limit $l\to\infty$ the classical value of the momentum is zero (since a classical excitation requires a large number of quanta). Thus, the particles must lie at the positions of their *static* classical equilibrium, which, for the Sutherland model, are evenly spaced on the circle. This gives a natural explanation to the fact that the system is integrable only when the lattice points are taken to be equidistant [11].

The above suggests a natural generalization: To every integrable system of particles with internal degrees of freedom corresponds an integrable lattice system through an appropriate "classical" limit. In particular, there should be a lattice system with inverse square interactions and the lattice points positioned at the equilibrium positions of the Calogero system. In the remaining of this paper we rigorously establish this fact.

Consider the N-body Calogero system with potential [13]

$$V=\sum_i \tfrac{1}{2} l^2\omega^2 x_i^2+\sum_{i<j}\frac{l^2}{x_{ij}^2} . \quad (8)$$

In order for this system to have a nontrivial classical equilibrium configuration at the $l\to\infty$ limit we took the strength of the harmonic oscillator potential to scale as l^2, else the particles will either collapse to the origin or fly away to infinity. Then, following Fowler and Minahan, we consider a system of particles with internal degrees of freedom with the Hamiltonian

$$H=\sum_{i<j}\frac{1}{x_{ij}^2}M_{ij} , \quad (9)$$

where the particle positions x_i are taken to minimize the above potential. The parameter ω can be absorbed into a rescaling of the particle positions x_i which results in a mere rescaling of the Hamiltonian (9). We put it therefore equal to 1, and the x_i satisfy

$$x_i-\sum_{j\neq i}\frac{2}{x_{ij}^3}=0 . \quad (10)$$

Consider then the operators

$$\pi_i=\sum_{j\neq i}\frac{i}{x_{ij}}M_{ij} . \quad (11)$$

These can be thought of as the large-l limit of the corresponding operators $\bar{\pi}_i$ defined in [12]

$$\bar{\pi}_i=p_i+l\pi_i . \quad (12)$$

Similarly, the Hamiltonian (9) can be thought of as the large-l limit of the full Hamiltonian

$$\bar{H}=\bar{H}_0+\sum_i \tfrac{1}{2} l^2 x_i^2 , \quad (13)$$

$$\bar{H}_0=\sum_i \tfrac{1}{2}\bar{\pi}_i^2=\sum_i \tfrac{1}{2}p_i^2-lH+\sum_{i<j}\frac{l^2}{x_{ij}^2} , \quad (14)$$

after dropping trivial nondynamical terms of order l^2. Since the operators $\bar{\pi}$ commute [12], we immediately obtain

$$[\pi_i,\pi_j]=0 . \quad (15)$$

The commutation properties of π_i with H can be calculated directly. All the labor can be saved, however, by taking the relation

$$[\bar{\pi}_i,\bar{H}_0]=0 \quad (16)$$

and expanding in powers of l. Since it holds for all l, each term must separately vanish. The term of order l^2 gives

$$\left[p_i,\sum_{i<j}\frac{1}{x_{ij}^2}\right]-[\pi_i,H]=0 \quad (17)$$

and we immediately obtain

$$[\pi_i,H]=\sum_{j\neq i}\frac{2i}{x_{ij}^3} . \quad (18)$$

Again, following [12], consider the operators

$$a_i^\dagger=\pi_i+ix_i,\quad a_i=\pi_i-ix_i , \quad (19)$$

$$h_i=a_i^\dagger a_i=\sum_i(\pi_i^2+x_i^2)-\sum_{j\neq i}M_{ij} . \quad (20)$$

The commutation relations of h_i can be directly deduced from the corresponding relations for $\bar{h}_i$ to be

$$[h_i,h_j]=-2(h_iM_{ij}-M_{ij}h_i) \quad (21)$$

and this means that the permutation symmetric quantities

$$I_n=\sum_i h_i^n \quad (22)$$

commute among themselves. The proof is by now standard [12] and will not be repeated here.

It remains to show that the I_n commute with H. To this end we have

$$[\pi_i^2,H]=\pi_i[\pi_i,H]+[\pi_i,H]\pi_i$$
$$=\pi_i\sum_{j\neq i}\frac{2i}{x_{ij}^3}+\sum_{j\neq i}\frac{2i}{x_{ij}^3}\pi_i \qquad (23)$$

and

$$[x_i^2,H]=\sum_{j\neq i}\frac{x_i^2-x_j^2}{x_{ij}^2}M_{ij}=\sum_{j\neq i}\frac{x_i+x_j}{x_{ij}}M_{ij}$$
$$=-ix_i\pi_i-i\pi_i x_i \,. \qquad (24)$$

Finally, H being permutation symmetric we have

$$[M_{ij},H]=0\,. \qquad (25)$$

Putting everything together we get

$$[h_i,H]=-i\left(x_i-\sum_{j\neq i}\frac{2}{x_{ij}^3}\right)\pi_i-i\pi_i\left(x_i-\sum_{j\neq i}\frac{2}{x_{ij}^3}\right). \qquad (26)$$

We see that the quantity appearing in the parenthesis is exactly the equation for classical equilibrium (10). Therefore, the quantities h_i, and consequently also I_n, will commute with the Hamiltonian if the x_i are chosen to correspond to the positions of Calogero particles at rest.

The remaining argument is as in [11]. The I_n will contain strings of operators M_{ij} which, when acting on totally symmetric (bosonic) states, become strings of operators σ_{ij} in the reverse order and those, in their turn, can be substituted by P_{ij} operators. These so-reduced operators $\tilde{I}_n$ then will constitute the commuting integrals of motion of a lattice system with Hamiltonian as in (9) but with the spin exchange operator P_{ij} appearing instead of the position exchange operator M_{ij}.

The integrals obtained above should contain the Hamiltonian itself, but, just as in the Haldane-Shastry model, they do so in a nontrivial way. In fact, since each h_i in our model contains two exchange operators, as opposed to only one in π_i for the previous model, the form of I_n is more complicated. The lowest integral I_1 is

$$I_1=\sum_i h_i=\sum_i x_i^2+\sum_{i\neq j}\frac{1}{x_{ij}^2}-\sum_{i\neq j}M_{ij}\,. \qquad (27)$$

Apart from a nondynamical term (which is twice the classical rest energy of the Calogero system) it consists only of a trivial exchange operator. The nontrivial conserved quantities are contained in higher I_n.

How different is our system from the Haldane-Shastry system? Conceivably, the different form of the interaction in (9) could conspire with the different lattice spacings to give something similar. It is easy to see, however, that this is not the case. Even the smallest nontrivial systems, $N=3$, are distant: The Haldane-Shastry system consists of three equal strength exchange interactions, while our model consists of three interactions with strengths at the ratio 4:4:1. Further, our system exhibits no analog of the (lattice) translation invariance of the Haldane-Shastry system. The dynamical properties of this system remain an interesting issue.

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