Invariants of the Haldane-Shastry SU(N) Chain

Michael Fowler[(a)] and Joseph A. Minahan[(b)]
Department of Physics, Jesse Beams Laboratory, University of Virginia, Charlottesville, Virginia 22901
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Using a formalism developed by Polychronakos, we explicitly construct a set of invariants of the motion for the Haldane-Shastry SU(N) chain.

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There have been several recent papers on the Haldane-Shastry model for spin chains and its SU(N) generalization [1–7]. This model is described by the Hamiltonian

$$H=\sum_{i<j}\frac{1}{\sin^2(\pi/L)(x_i-x_j)}P_{ij}\,, \tag{1}$$

where x_i are the positions of the spins, equally spaced around a ring, and P_{ij} is the operator that exchanges the spins at sites i and j. Haldane and Shastry found the wave functions for the antiferromagnetic ground state [1], showing it to be identical in form to the Sutherland ground state wave function for particles on a line with the inverse square potential [8]. These authors also found all possible energy levels for the system.

It would thus seem that the Haldane-Shastry model and its generalizations are integrable systems. If the model is integrable, there must exist a set of operators that commute among themselves and with the Hamiltonian. Inozemtsev found the first such nontrivial operator, one involving the exchange of three spins [2]. Haldane later found two others, a four-spin-exchange operator that commutes with both the Hamiltonian and with Inozemtsev's operator, and a more basic two-spin-exchange vector operator he refers to as the rapidity [9].

In this paper we explicitly show that the Haldane-Shastry model is integrable by constructing a complete set of operators that commute among themselves and with the Hamiltonian. These operators are very similar in structure to those used by Polychronakos [10] in his exchange operator approach to the Sutherland and Calogero models [8,11,12], and by Polychronakos and one of the authors in generalizations of these models [13].

The key to our approach is that we consider the system as a set of N bosons with internal degrees of freedom which sit on the N sites of the lattice, only allowing states with one particle per site, as in the infinite-U Hubbard model. The exchange terms making up the Hamiltonian provide both the kinetic energy, from hopping exchange of particles with different internal quantum states, and the potential energy. The new feature revealed by our approach is that there are very simple single *particle* (as opposed to single *site*) operators that commute with the Hamiltonian, analogous to those used by Polychronakos for the continuum case [10]. This makes it possible to construct a series of extensive conserved quantities.

To begin, let us assume that we have N bosonic particles sitting at different points on the circle. Let us further assume that the system propagates only by the particles exchanging their positions. Therefore, if the system starts with N particles on N different sites, the system will evolve with one particle on each of these same N sites. If the bosons had no other quantum numbers besides their positions, then this would be a trivial system. However, if the bosons have internal degrees of freedom then we will find a system with nontrivial dynamics.

Inspired by the work of Polychronakos [10], we define an operator π_i,

$$\pi_i=\sum_{j\neq i}\frac{z_j}{z_{ij}}M_{ij}\,, \tag{2}$$

where $z_i=\exp(2\pi i x_i/L)$, x_i are the particle positions, $z_{ij}=z_i-z_j$ and M_{ij} is the operator that exchanges the positions of particles i and j. M_{ij} is a Hermitian operator that satisfies the relations

$$\begin{aligned}&M_{ij}z_i=z_jM_{ij},\quad M_{ij}z_k=z_kM_{ij}\ \text{ if } i\neq k\neq j\,,\\ &M_{jik}\equiv M_{ij}M_{ik}=M_{jk}M_{ij}=M_{ik}M_{jk}\,.\end{aligned} \tag{3}$$

Using these relations it is straightforward to show that

$$[\pi_i,\pi_j]=M_{ij}\pi_i-\pi_iM_{ij}\,, \tag{4}$$

and therefore

$$[\pi_i^n,\pi_j]=M_{ij}\pi_i^n-\pi_i^nM_{ij}\,. \tag{5}$$

The operator π_i is very similar to an operator considered by Polychronakos, the only difference being that our operator does not contain an explicit kinetic term.

Next consider the operator $\tilde I_n$,

$$\tilde I_n=\sum_i\pi_i^n\,. \tag{6}$$

Computing the commutator of $\tilde I_n$ with $\tilde I_m$ we find

$$\begin{aligned}[\tilde I_n,\tilde I_m]&=\sum_{i,j}[\pi_i^n,\pi_j^m]=\sum_{i,j}\sum_{a=0}^{m-1}\pi_j^a[\pi_i^n,\pi_j]\pi_j^{m-a-1}\\ &=-\sum_{i,j}\left(\sum_{a=0}^{m-1}-\sum_{a=n}^{m+n-1}\right)\pi_j^aM_{ij}\pi_j^{m+n-a-1}\,.\end{aligned} \tag{7}$$

Explicitly antisymmetrizing in m and n then gives

$$[\tilde{I}_n,\tilde{I}_m] = -\sum_{i,j}\left(\sum_{a=0}^{m-1} - \sum_{a=n}^{m+n-1} - \sum_{a=0}^{n-1} + \sum_{a=m}^{m+n-1}\right) h_j^a \times M_{ij} h_j^{m+n-a-1} = 0 . \tag{8}$$

Note that the commutation of these operators does not need the spacing between the sites to be equidistant.

We next relate the $\tilde{I}_n$ operators to corresponding operators in the Haldane-Shastry model. Operators in the Haldane-Shastry model involve the exchange of spins at particular sites on the lattice. Our operators involve the exchange of positions of particles that live on each site. But we can invoke the fact that the particles are identical to relate the two sets of operators [13]. Let us define an operator σ_{ij} that exchanges the spins of two particles, but not their positions. If the particles are identical, then the product $\sigma_{ij}M_{ij}$ acting on a symmetric wave function is unity. Moreover, since σ_{ij} acts on spins and M_{ij} acts on the positions, the two operators commute with each other. Hence, if we have a chain of M operators acting on a symmetric state, we can substitute for it a chain of spin exchange operators. For example, we can make the substitution on the following product of operators:

$$M_{ij}M_{jk}M_{km}|\psi\rangle = M_{ij}M_{jk}\sigma_{km}|\psi\rangle = \sigma_{km}M_{ij}\sigma_{jk}|\psi\rangle = \sigma_{km}\sigma_{jk}\sigma_{ij}|\psi\rangle . \tag{9}$$

This product of M_{ij} operators is the cyclic exchange operator for four particles. The corresponding spin product is the cyclic exchange of spins in the opposite direction.

Finally, to relate these spin operators of *particles* to spin operators at *sites*, we note that there is always one particle at every site. Hence any operator that contains symmetric sums over all particles can be substituted with an operator that sums over all sites, and where the spin exchange operator for *particles* σ_{ij}, is replaced by the exchange operator for *sites* P_{ij}. In particular, the operators $\tilde{I}_n$ are now replaced with new operators I_n. The commutation relations of the I_n operators will be the same as the commutation relations for the original operators.

To complete the proof of integrability, we consider the operator $\tilde{H}$,

$$\tilde{H} = \sum_j \tilde{H}_j ,$$
$$\tilde{H}_j = \sum_{k\neq j} \frac{1}{\sin^2(\pi/L)(x_j-x_k)} M_{jk} = -4\sum_{k\neq j}\frac{z_k z_j}{(z_{kj})^2} M_{jk} . \tag{10}$$

We now show that all π_i commute with $\tilde{H}$, if the sites are equally spaced. Consider first the commutator of π_i with $\tilde{H}_j$ if $i\neq j$. Using the relations in (3), we find that

$$\begin{aligned}
[\pi_i,\tilde{H}_j] &= \sum_{\substack{k\neq i\\ l\neq j}}\left[\frac{z_k}{z_{ik}}M_{ik},\frac{-4z_jz_l}{(z_{jl})^2}M_{jl}\right] \\
&= -4\sum_{k\neq i,j}\left\{\left[\frac{z_k}{z_{ik}}M_{ik},\frac{z_jz_i}{(z_{ij})^2}M_{ij}\right]+\left[\frac{z_j}{z_{ij}}M_{ik},\frac{z_jz_i}{(z_{ij})^2}M_{ij}\right]+\left[\frac{z_k}{z_{ik}}M_{ik},\frac{z_jz_k}{(z_{jk})^2}M_{jk}\right]\right\} \\
&= -4\sum_{k\neq i,j}\left(\frac{z_k}{z_{ik}}\frac{z_jz_k}{(z_{jk})^2}+\frac{z_j}{z_{ij}}\frac{z_iz_k}{(z_{ik})^2}-\frac{z_j}{z_{ij}}\frac{z_jz_k}{(z_{jk})^2}\right)M_{ijk} - 4\sum_{k\neq i,j}\left(-\frac{z_k}{z_{ik}}\frac{z_jz_k}{(z_{jk})^2}-\frac{z_k}{z_{jk}}\frac{z_iz_j}{(z_{ij})^2}-\frac{z_k}{z_{ik}}\frac{z_jz_i}{(z_{ij})^2}\right)M_{jik} \\
&= -4\sum_{k\neq i,j}\left\{-\frac{z_iz_jz_k}{z_{jk}(z_{ik})^2}M_{ijk}-\frac{z_jz_kz_j}{z_{ij}(z_{jk})^2}M_{jik}\right\} .
\end{aligned} \tag{11}$$

Next consider the commutator of π_i with $\tilde{H}_i$. We find

$$\begin{aligned}
[\pi_i,\tilde{H}_i] &= -4\sum_{\substack{k\neq i\\ l\neq i}}\left[\frac{z_k}{z_{ik}}M_{ik},\frac{z_iz_l}{(z_{il})^2}M_{il}\right] \\
&= -4\sum_{k\neq i}\left[\frac{z_k}{z_{ik}}M_{ik},\frac{z_iz_k}{(z_{ik})^2}M_{ik}\right] - 4\sum_{\substack{k,l\neq i\\ k\neq l}}\left[\frac{z_k}{z_{ik}}M_{ik},\frac{z_iz_l}{(z_{il})^2}M_{il}\right] \\
&= -4\sum_{k\neq i}\frac{z_i+z_k}{z_{ik}}\frac{z_iz_k}{(z_{ik})^2} - 4\sum_{\substack{k,l\neq i\\ k\neq l}}\left(\frac{z_kz_kz_l}{z_{ik}(z_{kl})^2}M_{kil}-\frac{z_iz_lz_k}{z_{lk}(z_{il})^2}M_{ikl}\right) .
\end{aligned} \tag{12}$$

Summing over j in (11) and adding the expression in (12), we are left with

$$[\pi_i,\tilde{H}] = -4\sum_{k\neq i}\frac{z_i+z_k}{z_{ik}}\frac{z_iz_k}{(z_{ik})^2} . \tag{13}$$

In general, this expression is not zero. However, if we assume that the sites are equally spaced, then by translational invariance and the antisymmetry of the summand, the sum is zero.

Since all π_i commute with $\tilde{H}$ then clearly, all $\tilde{I}_n$ must commute with $\tilde{H}$ as well. We may now perform the substitution of H for $\tilde{H}$ in the same way that I_n is substituted for $\tilde{I}_n$. Hence all I_n commute with H and therefore the system is integrable.

Having established that the I_n form a commuting set of operators, we now examine some of these operators explicitly. The first such operator, I_1, is found from $\tilde{I}_1$, which is given by

$$\tilde{I}_1=\sum_{i\neq j}\frac{z_j}{z_{ij}}M_{ij}=-\frac{1}{2}\sum_{i\neq j}\frac{z_{ij}}{z_{ij}}M_{ij}\,. \tag{14}$$

Thus I_1 satisfies

$$I_1=-\frac{1}{2}\sum_{i\neq j}P_{ij}=-\frac{N(N-4)}{4}-(\mathbf{S}\cdot\mathbf{S})\,, \tag{15}$$

where $\mathbf{S}$ is the total spin of the system. This operator trivially commutes with the Hamiltonian.

A more interesting operator is I_2, where $\tilde{I}_2$ is given by

$$\tilde{I}_2=\sum_i\sum_{\substack{j\neq i\\k\neq i}}\frac{z_j}{z_{ij}}M_{ij}\frac{z_k}{z_{ik}}M_{ik}=\sum_{i\neq j\neq k\neq i}\frac{z_j}{z_{ij}}\frac{z_k}{z_{jk}}M_{ikj}+\sum_{i\neq j}\frac{z_iz_j}{(z_{ij})^2}\,. \tag{16}$$

The last term is just a constant. Symmetrizing the sum over i, j, and k, we find that $\tilde{I}_2$ reduces to

$$\tilde{I}_2=\frac{1}{2}\sum_{i\neq j\neq k\neq i}\frac{z_i+z_j}{z_{ij}}M_{ijk}+\frac{1}{6}\sum_{i\neq j\neq k\neq i}M_{ijk}-\frac{1}{12}(N^2-1)\,. \tag{17}$$

Hence we have

$$I_2=-\frac{1}{2}\sum_{i\neq j\neq k\neq i}\frac{z_i+z_j}{z_{ij}}P_{ijk}+\frac{1}{6}\sum_{i\neq j\neq k\neq i}P_{ijk}-\frac{1}{12}(N^2-1)\,. \tag{18}$$

The second term is a trivial exchange operator that commutes with the Hamiltonian and the other I_n, therefore the first term must do so as well. To demonstrate the significance of this operator, let us specialize to the case of SU(2). The antisymmetric piece of P_{ijk} is given by $-i(\boldsymbol{\sigma}_i\times\boldsymbol{\sigma}_j)\cdot\boldsymbol{\sigma}_k$. Since the first term in (18) has no z_k dependence, it can be reexpressed as

$$\frac{i}{2}\sum_{i\neq j}\frac{z_i+z_j}{z_{ij}}(\boldsymbol{\sigma}_i\times\boldsymbol{\sigma}_j)\cdot\mathbf{S}=\boldsymbol{\Lambda}\cdot\mathbf{S}\,. \tag{19}$$

Since the Hamiltonian is isotropic in the total spin, then each component of $\boldsymbol{\Lambda}$ must commute with H. This operator is the rapidity operator defined by Haldane [8]. Let us act with this operator on the one magnon state, described by the wave function

$$|\psi\rangle=\sum_n\sigma_n^+e^{ikn}|0\rangle\,, \tag{20}$$

where $|0\rangle$ is the all spins down state. Acting on this state with Λ_z gives

$$\frac{1}{2}\sum_{i\neq j}\frac{z_i+z_j}{z_{ij}}\frac{1}{2}(\sigma_i^-\sigma_j^+-\sigma_i^+\sigma_j^-)\sum_n(z_n)^k\sigma_n^+|0\rangle=\frac{1}{2}\sum_{i\neq j}\frac{z_i+z_j}{z_{ij}}\left(\frac{z_j}{z_i}\right)^k|\psi\rangle$$
$$=\frac{1}{2}N\sum_{j\neq 0}\frac{1+z_j}{1-z_j}(z_j)^k|\psi\rangle=N(k-N/2)|\psi\rangle\,. \tag{21}$$

Hence the z component of $\boldsymbol{\Lambda}$ acts like a momentum operator.

For higher I_n, one can show that the leading term is of the form

$$I_n\propto\sum_{i_1\neq\ldots,i_{n+1}}\frac{z_{i_1}+z_{i_2}}{z_{i_1}-z_{i_2}}\frac{z_{i_2}+z_{i_3}}{z_{i_2}-z_{i_3}}\cdots\frac{z_{i_{n-1}}+z_{i_n}}{z_{i_{n-1}}-z_{i_n}}P_{i_1,\ldots,i_{n+1}}\,. \tag{22}$$

This is basically a generalization of the rapidity operator and is not quite of the Inozemtsev-Haldane form for invariants. Since the leading term in I_n contains an n-spin exchange term, it must be independent of all I_m, $m<n\leq N$, since I_m will not have such a term.

Presumably, the Hamiltonian as well as the Inozemtsev-Haldane invariants lurk within our operators, but they fit in a nontrivial way. For instance, after a particularly tedious calculation one can show that I_3 is given by

$$I_3=\boldsymbol{\Lambda}_2\cdot\mathbf{S}-\frac{N-3}{4}\boldsymbol{\Lambda}\cdot\mathbf{S}-\boldsymbol{\Lambda}\cdot\boldsymbol{\Lambda}+\frac{3}{4}H-\frac{N^2+3N+20}{24}\sum_{i\neq j}P_{ij}+\frac{3N-14}{24}\sum_{i\neq j\neq k}P_{ijk}-\frac{1}{8}\sum_{i\neq j\neq k\neq l}P_{ijkl}+C\,, \tag{23}$$

where $\boldsymbol{\Lambda}_2$ is

$$\boldsymbol{\Lambda}_2=-\frac{1}{2}\sum_{i\neq j\neq k}\frac{z_i+z_j}{z_{ij}}\frac{z_j+z_k}{z_{jk}}P_{ijk}\boldsymbol{\sigma}_k\,. \tag{24}$$

The Hamiltonian explicitly appears in I_3, but one can also show that the z component of $\boldsymbol{\Lambda}_2$ acting on the one magnon state satisfies

$$\Lambda_{2z}|\psi\rangle=-\frac{1}{4}\left(H-(N-1)\sum_{i\neq j}P_{ij}+\frac{1}{3}(N^2-5)\right)|\psi\rangle\,. \tag{25}$$

The Hamiltonian and $\boldsymbol{\Lambda}\cdot\boldsymbol{\Lambda}$ acting on the one magnon state lead to terms quadratic in the momentum, hence all terms in I_3 are basically equivalent to terms containing the Hamiltonian or the rapidity, at least when acting on one magnon states. Likewise I_4 will contain Inozemtsev's operator [2],

$$\sum_{i\neq j\neq k}\frac{z_iz_jz_k}{z_{ij}z_{jk}z_{ki}}P_{ijk}\,, \tag{26}$$

and other operators that lead to terms cubic in the momentum when acting on a single magnon.

In conclusion, we have shown that the Haldane-Shastry SU(N) chain is integrable by explicitly constructing a set of independent invariants of the motion. For the discrete case considered here, the Hamiltonian appears in the third level of invariants. This contrasts to the Sutherland continuum model, where the Hamiltonian first appears in the second level of invariants. In general, I_n acting on one magnon states will give $n-1$ powers of the momentum. Hence the I_n are like derivative operators, although there is one less derivative than in the continuum case.

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(a)Electronic address: mfli@landau1.phys.virginia.edu
(b)Electronic address: minahan@gomez.phys.virginia.edu

[1] F. D. M. Haldane, Phys. Rev. Lett. **60**, 635 (1988); B. S. Shastry, Phys. Rev. Lett. **60**, 639 (1988).
[2] V. I. Inozemtsev, J. Stat. Phys. **59**, 1143 (1989).
[3] F. D. M. Haldane, Phys. Rev. Lett. **66**, 1529 (1991).
[4] H. Kiwata and Y. Akutsu, J. Phys. Soc. Jpn. **61**, 1441 (1992).
[5] N. Kawakami, Phys. Rev. B **46**, 1005 (1992); **46**, 3191 (1992).
[6] B. S. Shastry, Phys. Rev. Lett. **69**, 164 (1992).
[7] Z. N. C. Ha and F. D. M. Haldane, Phys. Rev. B **46**, 9359 (1992).
[8] B. Sutherland, Phys. Rev. A **4**, 2019 (1971); **5**, 1372 (1972); Phys. Rev. Lett. **34**, 1083 (1975).
[9] F. D. M. Haldane (private communication).
[10] A. P. Polychronakos, Phys. Rev. Lett. **69**, 703 (1992).
[11] F. Calogero, J. Math. Phys. **10**, 2191 (1969); **10**, 2197 (1969); **12**, 419 (1971).
[12] J. Moser, Adv. Math. **16**, 1 (1975); F. Calogero, Lett. Nuovo Cimento **13**, 411 (1975); F. Calogero and C. Marchioro, Lett. Nuovo Cimento **13**, 383 (1975).
[13] J. A. Minahan and A. P. Polychronakos, University of Virginia Report No. UVA-HET-92-04, Columbia University Report No. CU-TP-566 (to be published).