Quantum Trajectory Theory for Cascaded Open Systems

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The quantum trajectory theory of an open quantum system driven by a photoemissive source is formulated. The formalism is illustrated by applying it to photon scattering from an atom driven by strongly focused coherent light.

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Research on the generation of nonclassical light has been carried out with considerable success for several years. There has been little work, however, on using nonclassical light to excite a second quantum system. Most experiments are concerned either with directly measuring the nonclassical characteristics of a source or using these characteristics indirectly to illustrate some quantum effect. A common feature is that there is, at most, a linear transformation of fields between the source and the detector. The theory of such experiments need only calculate some low-order correlation function of the fields.

What work there is on the interaction of a quantum system with nonclassical light also concerns situations in which the light is characterized by law-order correlation functions. Much of it is an extension of Gardiner's treatment of a two-state atom interacting with broadband squeezed light [1], where the first-order correlation function characterizes the field. Beyond this, a considerable amount of work exists dealing with interactions that take place inside cavities. However, here a direct Hamiltonian coupling is used, which treats the interaction in a time-symmetric way. I am concerned with *open* system interactions: a quantum source A emits photons and a second quantum system B reacts to the emitted photons. No general approach exists to solve such problems, aside from that of writing down an infinite set of Heisenberg equations which can only be solved for the simplest, linear coupling, examples. In this Letter I propose a tractable approach based on quantum trajectory theory [2–5]. The quantum trajectories are constructed from a master equation derived using reservoir theory. This equation agrees with an equation obtained in a less direct way by Kolobov and Sokolov [6].

It is natural to divide the problem into two parts: First compute the properties of the field radiated by A; then compute the response of B in terms of the known properties of A. There is a difficulty here, however. In general an infinite number of correlation functions are needed to characterize the field radiated by A. In semiclassical theory this might be handled by generating stochastic realizations of the radiated field and computing results by numerical simulation. But the correlation functions for nonclassical light cannot be produced by a stochastic field. It is better, then, not to divide the problem into two parts. My description is made in terms of a stochastic wave function for the composite system $A\oplus B$. To obtain the broken time symmetry I allow the interaction between A and B to be mediated by a reservoir R and use the Born-Markoff approximation. Figure 1 illustrates a simple version of the source and driven system, where I assume that only one mode of each cavity need be considered. The cavities have three perfectly reflecting mirrors and one mirror with transmission coefficient $T\ll 1$. Hamiltonians $\hat{H}_A$ and $\hat{H}_B$ describe the free cavity modes and any interactions that take place inside the cavities. $\hat{H}_R$ is the free Hamiltonian of a traveling-wave reservoir which couples the cavities in one direction only. The fields $\hat{\mathcal{E}}(0)$ and $\hat{\mathcal{E}}(l)$ that couple to the cavities are written in photon flux units.

The complete Hamiltonian for $A\oplus B\oplus R$ is

$$\hat{H}=\hat{H}_A+\hat{H}_B+\hat{H}_R+\hat{H}_{AR}+\hat{H}_{BR}\,, \tag{1}$$

with

$$\begin{aligned}\hat{H}_{AR}&=i\hbar(2\kappa_A)^{1/2}[\hat{a}\hat{\mathcal{E}}^\dagger(0)-\text{H.c.}]\,,\\ \hat{H}_{BR}&=i\hbar(2\kappa_B)^{1/2}[\hat{b}\hat{\mathcal{E}}^\dagger(l)-\text{H.c.}]\,,\end{aligned} \tag{2}$$

where κ_A and κ_B are the cavity linewidths, and $\hat{a}$ and $\hat{b}$ are annihilation operators for the cavity modes. $\hat{H}$ describes two systems interacting with the same reservoir. Problems of this type are not new. Dicke superradiance, for example, involves many atoms interacting with the same reservoir; the common reservoir interaction produces a coupling of the atoms through the relaxation terms in the master equation. Equation (2) is different,

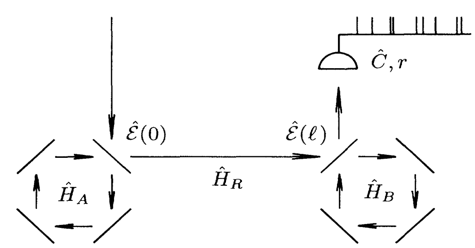

FIG. 1. Open quantum system B cascaded with a quantum source A.

however, because A and B couple to the reservoir at different spatial locations. Usually spatially separated reservoir fields are treated as statistically independent. Of course, this cannot be done for a geometry like that in Fig. 1 where the output from A appears, after a delay $\tau = l/c$, in the field that couples to B. We can, in fact, eliminate the spatial separation by using the Born-Markoff approximation in the Heisenberg picture to relate $\hat{\mathscr{E}}(l)$ to $\hat{\mathscr{E}}(0)$:

$$\hat{U}_A(\tau)\hat{\mathscr{E}}(l)\hat{U}_A^{-1}(\tau) = \hat{\mathscr{E}}(0) + \tfrac{1}{2}(2\kappa_A)^{1/2}\hat{a}\,, \tag{3}$$

where

$$\hat{U}_A(\tau) = \exp[(i/\hbar)(\hat{H}_A + \hat{H}_R + \hat{H}_{AR})\tau]\,. \tag{4}$$

Equation (3) states that the field at l is the time-retarded field at 0, including a component radiated by A. Then if $\chi(t)$ is the density operator that evolves according to the Liouville equation with Hamiltonian (1), we may define the source-retarded density operator

$$\chi'(t) = \hat{U}_A(\tau)\chi(t)\hat{U}_A^{-1}(\tau)\,, \tag{5}$$

and show, using Eqs. (1)-(4), that $\chi'(t)$ satisfies the Liouville equation with Hamiltonian

$$\hat{H}' = \hat{H}_S + \hat{H}_R + \hat{H}_{SR}\,, \tag{6}$$

where

$$\begin{aligned} \hat{H}_S &= \hat{H}_A + \hat{H}_B + i\hbar(\kappa_A\kappa_B)^{1/2}(\hat{a}^\dagger\hat{b} - \mathrm{H.c.})\,, \\ \hat{H}_{\mathrm{SR}} &= i\hbar\{[(2\kappa_A)^{1/2}\hat{a} + (2\kappa_B)^{1/2}\hat{b}]\hat{\mathscr{E}}^\dagger(0) - \mathrm{H.c.}\}\,. \end{aligned} \tag{7}$$

Notice that a and B now couple to the reservoir at the same spatial location. They also couple directly, with coupling constant $(\kappa_A\kappa_B)^{1/2}$.

The derivation of the master equation corresponding to $\hat{H}'$ is standard. I define the source-retarded reduced density operator $\rho' = \mathrm{tr}_R(\chi')$ and find

$$\dot{\rho}' = (1/i\hbar)[\hat{H}_S, \rho'] + \hat{C}\rho'\hat{C}^\dagger - \tfrac{1}{2}\hat{C}^\dagger\hat{C}\rho' - \tfrac{1}{2}\rho'\hat{C}^\dagger\hat{C}\,, \tag{8}$$

with

$$\hat{C} = (2\kappa_A)^{1/2}\hat{a} + (2\kappa_B)^{1/2}\hat{b}\,. \tag{9}$$

$\rho'(t)$ is all that is needed to calculate results for B. If, however, the unretarded density operator is required, from Eq. (5) we have $\rho(t) = \exp(\mathcal{L}_A\tau)\rho'(t)$, where $\dot{\rho}_A = \mathcal{L}_A\rho_A$ is the master equation for A alone [obtained from Eq. (8) by tracing over B].

The master equation (8) can be used directly to solve problems involving a quantum system interacting with nonclassical light. For analytical calculations it will often be the best place to start. On the other hand, the quantum trajectories defined by this master equation also provide a powerful computational method. More importantly, they clarify the physical interpretation. In quantum trajectory theory $\rho'(t)$ is replaced by an ensemble of stochastic wave functions $|\psi_c(t)\rangle$ which describe the state of the system *conditioned* on the realization of a particular history of signals at idealized detectors that monitor the radiated fields. In this instance there is one detector, as depicted in Fig. 1. It sees the superposition of fields that enters the definition of the operator $\hat{C}$ [Eq. (9)]; these fields cannot be monitored individually, in principle, without upsetting the coupling between A and B. The evolution of $|\psi_c(t)\rangle$ is given in terms of the unnormalized wave function $|\bar{\psi}_c(t)\rangle$. Following the prescription for tracing the statistics of photoelectron emissions back to the source [2] I find that, between emissions, $|\bar{\psi}_c(t)\rangle$ satisfies the Schrödinger equation $|\dot{\bar{\psi}}_c\rangle = (1/i\hbar)\hat{\mathcal{H}}|\bar{\psi}_c\rangle$, with non-Hermitian Hamiltonian

$$\hat{\mathcal{H}} = \hat{H}_A + \hat{H}_B - i\hbar[\kappa_A\hat{a}^\dagger\hat{a} + \kappa_B\hat{b}^\dagger\hat{b} + 2(\kappa_A\kappa_B)^{1/2}\hat{a}\hat{b}^\dagger]\,. \tag{10}$$

The emission times are determined in a Monte Carlo fashion using the rate function $r(t) = \langle\psi_c(t)|\hat{C}^\dagger\hat{C}|\psi_c(t)\rangle$, and each emission is accompanied by the wave-function collapse $|\bar{\psi}_c(t)\rangle \to \hat{C}|\bar{\psi}_c(t)\rangle$.

I emphasize that there is no approximation in passing from Eq. (8) to Eq. (10), only a decomposition of the mixed state $\rho'(t)$ into an ensemble of pure states $|\psi_c(t)\rangle$. The result is a formulation that expresses the coupling between A and B in a simple and natural way. First, the non-Hermitian Hamiltonian (10) includes an interaction that annihilates photons from A and creates them in B; the reverse process does not occur, as we might expect for the coupling between open systems. The asymmetric interaction arises from a cancellation of terms contributed by the reversible and irreversible parts of Eq. (8). Second, A and B are coupled through the collapse operator $\hat{C}$ because detected photons cannot be associated with photon emissions from either A or B separately.

I illustrate the theory by applying it to a two-state atom driven on resonance by strongly focused coherent light (Fig. 2). The source in this case is not nonclassical, but the example is well suited to treatment by the new theory. The source A consists of a laser cavity with output coupling rate $2K$, radiating coherent light with photon flux $\mathcal{R}$. The coherent light is focused to a spot size on the order of an atomic absorption cross section. The quantum system B is a two-state atom placed at the focus of the coherent light.

The situation differs from that illustrated in Fig. 1 since the atom couples to modes other than those carrying the incident light. Thus, in Fig. 2 the 4π modes coupling to the atom are divided between four channels—two labeled by Γ and two by $\bar{\Gamma} = 1 - \Gamma$; Γ is the spontaneous emission rate into the solid angle subtended by the source. The incident light occupies one channel, and superposed with forward scattering, ultimately falls on the detector F. Backwards and sideways scattered photons appear in the remaining three channels (with vacuum inputs) which terminate at the detectors $\bar{F}$. Time is measured in units of twice the atomic lifetime and therefore

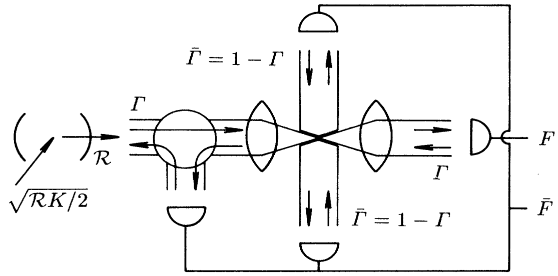

FIG. 2. Input-output channels for an atom driven by coherent light.

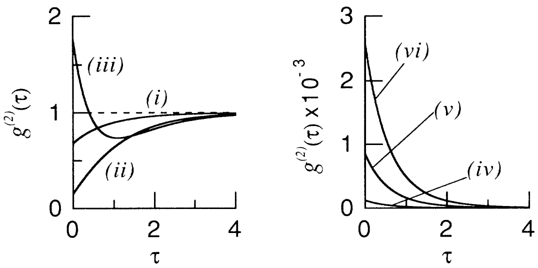

FIG. 3. Intensity correlation function at detector F for $\mathcal{R}=0.01$ and (i) $\Gamma=0.4$; (ii) $\Gamma=0.5$; (iii) $\Gamma=0.6$; (iv) $\Gamma=0.8$; (v) $\Gamma=0.9$; (vi) $\Gamma=1.0$.

all rates are dimensionless numbers; $2\Gamma+2\bar{\Gamma}=2$ is the total spontaneous emission rate and $0\le\Gamma\le 1$.

For this example the non-Hermitian Hamiltonian is

$$\hat{\mathcal{H}}=i\hbar[\sqrt{\mathcal{R}K/2}(\hat{a}^\dagger-\hat{a})-K\hat{a}^\dagger\hat{a} -\hat{\sigma}_+\hat{\sigma}_- -\sqrt{2K\Gamma}\hat{a}\hat{\sigma}_+], \tag{11}$$

where the first term on the right-hand side represents a classical current driving the laser mode; $\hat{\sigma}_+$ and $\hat{\sigma}_-$ raise and lower the atom between states $|-\rangle$ (lower) and $|+\rangle$ (upper). There are now two kinds of collapses occurring at rates $R_F(t)=\langle\psi_c(t)|\hat{C}_F^\dagger\hat{C}_F|\psi_c(t)\rangle$ and $R_{\bar{F}}(t)=\langle\psi_c(t)|\hat{C}_{\bar{F}}^\dagger\hat{C}_{\bar{F}}|\psi_c(t)\rangle$, defined by the collapse operators

$$\hat{C}_F=\sqrt{2K}\hat{a}+\sqrt{\Gamma}\hat{\sigma}_-,\quad \hat{C}_{\bar{F}}=\sqrt{2-\Gamma}\hat{\sigma}_- . \tag{12}$$

If the cavity mode is initially in the vacuum state, the conditioned wave function factorizes in the form $|\psi_c(t)\rangle=|\alpha(t)\rangle|A_c(t)\rangle$, where $|\alpha(t)\rangle$ is a coherent state and $|A_c(t)\rangle$ is the state of the atom. After a short time $\alpha(t)\to\alpha_{ss}=\sqrt{\mathcal{R}/2K}$. Then the quantum trajectory for the atom is governed by the Schrödinger equation and collapse operators

$$|\dot{\bar{A}}_c\rangle=-(\hat{\sigma}_+\hat{\sigma}_-+\sqrt{\mathcal{R}\Gamma}\hat{\sigma}_+)|\bar{A}_c\rangle, \tag{13}$$

$$\hat{C}_F=\sqrt{\mathcal{R}}+\sqrt{\Gamma}\hat{\sigma}_-,\quad \hat{C}_{\bar{F}}=\sqrt{2-\Gamma}\hat{\sigma}_- . \tag{14}$$

Equations (13) and (14) are equivalent to those for an atom *inside* a coherently driven cavity in the bad-cavity limit [7]; 2Γ and $2\bar{\Gamma}$ correspond to the spontaneous emission rates into the cavity mode and out the sides of the cavity, respectively. In the low-photon-flux limit, forward scattering in the cavity system is known to be antibunched [7,8]. The intensity correlation function showing this antibunching is plotted in Fig. 3. The figure also shows that antibunching is replaced by extreme photon bunching as Γ is increased. I focus here on this extreme photon bunching. What do the quantum trajectories say about this?

In the low-photon-flux limit the solution to Eq. (13) generally reaches steady state between collapses, with steady-state wave function

$$|A_c\rangle=(1+\mathcal{R}\Gamma)^{-1/2}(|-\rangle-\sqrt{\mathcal{R}\Gamma}|+\rangle). \tag{15}$$

Using Eqs. (14), the corresponding photon detection rates (photon fluxes) are

$$R_F=\mathcal{R}(1+\mathcal{R}\Gamma)^{-1}(\bar{\Gamma}^2+\mathcal{R}\Gamma),\qquad R_{\bar{F}}=\mathcal{R}-R_F . \tag{16}$$

Consider the case $\bar{\Gamma}=0$ which produces the largest bunching effect. In this case the incident light is focused within the atomic absorption cross section and we might expect a weak incident beam to be completely absorbed (reflected). Indeed, the transmitted photon flux is very small—$R_F\sim\mathcal{R}^2$ rather than $R_F\sim\mathcal{R}$. However, it is not zero; a few photons are transmitted. To understand why, and why these photons are highly bunched, we consider the wave-function collapse that accompanies the detection of a photon in transmission. Applying $\hat{C}_F$ to Eq. (15) gives

$$|A_c\rangle=(\bar{\Gamma}^2+\mathcal{R}\Gamma)^{-1/2}(\bar{\Gamma}|-\rangle-\sqrt{\mathcal{R}\Gamma}|+\rangle), \tag{17}$$

and the new detection rates

$$R_{F|F}=\mathcal{R}(\bar{\Gamma}^2+\mathcal{R}\Gamma)^{-1}[(\bar{\Gamma}-\Gamma)^2+\mathcal{R}\Gamma],\qquad R_{\bar{F}|F}=\mathcal{R}-R_F+\mathcal{R}(\bar{\Gamma}^2+\mathcal{R}\Gamma)^{-1}2\Gamma^2 . \tag{18}$$

For $\bar{\Gamma}=0$ the forward photon flux is now $\Gamma=1$, a change $R_{F|F}/R_F\sim 1/\mathcal{R}^2$. This huge increase in flux produces the extreme value of $g^{(2)}(0)$. The increase is explained by the collapse of the atomic wave function. If $\bar{\Gamma}=0$ and $\mathcal{R}\ll 1$, the atom collapses from near its ground state [Eq. (15)] *to its excited state* [Eq. (17)]. The rate $R_{F|F}=\Gamma=1$ is the forward spontaneous emission rate from the excited state. To explain the collapse, I note that the atom is capable of reflecting all the incident photons so long as it can deal with them one at a time. If, however, two photons arrive within an atomic lifetime (approximately), one photon can slip through while the atom is busy with the other. Thus, after the detection of a photon in transmission the atom collapses to its excited state, indicating that it had to let one photon by because it had just absorbed another.

For some incident photon pairs the second photon is

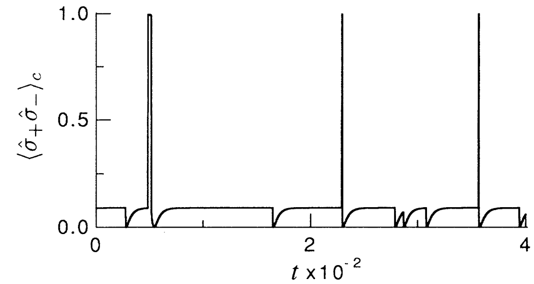

FIG. 4. Sample quantum trajectory for $\mathcal{R}=0.1$ and $\Gamma=1.0$.

detected in transmission after the first is detected in reflection. In this case $\hat{C}_{\bar{F}}$ collapses the atom to its ground state (normal photon antibunching) which increases the forwards flux by $R_{F|\bar{F}}/R_F \sim 1/R$.

The scenario behind the extreme photon bunching is illustrated by the trajectory in Fig. 4 where the conditioned excited state probability is plotted as a function of time ($\mathcal{R}$ is 10 times larger than in Fig. 3). The figure shows six backward scattering events, each accompanied by the signature of normal photon antibunching, and three events involving photon transmission. In the first of the three events both photons of a closely spaced pair are detected in the forward direction; in the second and third, the first photon of a pair is detected in the forward direction and the second is detected in the backward direction.

To conclude let me answer a question that might seem, superficially, to raise doubts about the new theory. It is usual to model the interaction between an atom and coherent light in a time-symmetric way, with an interaction Hamiltonian that can raise *and* lower the atom. $\hat{\mathcal{H}}$ [Eq. (11)] only includes the raising part; it is reasonable, then, to ask: How is it possible for a Rabi oscillation to occur? The answer lies in the collapse operator $\hat{C}_F$ [Eq. (14)]. This accounts for transitions that lower the atom while a photon is emitted into the coherent beam that excites the atom. Consider the limit $\mathcal{R}\rightarrow\infty$, $\Gamma\rightarrow 0$, with $\sqrt{R\Gamma}=\Omega/2$, where Ω is the Rabi frequency. In the approach to the limit, $\hat{C}_F$ will be applied a large number of times, $\Delta n \approx \mathcal{R}\Delta t + \sqrt{\mathcal{R}}\Delta W$, during a time interval Δt; ΔW is a Wiener increment. The change in $|\bar{A}_c\rangle$ during Δt, including the effect of the Δn collapses is

$$\Delta|\bar{A}_c\rangle \approx -[(\hat{\sigma}_+\hat{\sigma}_- + \sqrt{\mathcal{R}\Gamma}\hat{\sigma}_+)\Delta t - \Delta n\sqrt{\Gamma/\mathcal{R}}\hat{\sigma}_-]|\bar{A}_c\rangle . \tag{19}$$

In the limit, between the emissions generated by $\hat{C}_{\bar{F}}$ the unnormalized conditioned atomic wave function obeys the equation

$$|\dot{\bar{A}}_c\rangle = -[\hat{\sigma}_+\hat{\sigma}_- + (\Omega/2)(\hat{\sigma}_+ - \hat{\sigma}_-)]|\bar{A}_c\rangle . \tag{20}$$

Now the symmetric interaction needed to produce the Rabi oscillation appears explicitly.

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