Driving a Quantum System with the Output Field From Another Driven Quantum System

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Quantum Langevin equations and a master equation are derived for a two-atom system in which the first atom is driven by coherent field, and the fluorescent light used to drive a second atom. We show that the light beams from both atoms are antibunched, and that they are mutually anticorrelated.

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The development of the theory of nonclassical states of light [1–3] and the fact that it is now possible to produce reliable sources of both antibunched [4] and squeezed [5] light has led to the consideration of how one should describe the driving of an optical system by a nonclassical light field. This was first done by Gardiner [6] for the special situation of incoming squeezed white noise, and subsequently methods based on the adjoint equation [7] were used to describe the driving of a single atom with Gaussian, but nonwhite, squeezed light [8–10], which is a very good model of the light produced by a degenerate parametric oscillator.

It is ironic that the first kind of nonclassical light studied was antibunched light [4, 11], but it has so far been unknown how to treat the problem of driving a system with antibunched light, although the related problem of a laser with a sub-Poissonian pump has been treated in [12]. This is particularly interesting, since antibunching is a property of fourth order correlation functions, while squeezing is a property of second order correlation functions. In any Markovian approximation, only second order correlations are important, so that any fourth order correlation effects will not be noticeable in this degree of approximation. Thus, we can expect significant non-Markovian effects if a system is driven by antibunched light.

The major problem here is how one writes a physically acceptable formalism which allows coupling from atom 1 to atom 2 *without* allowing coupling in the reverse direction. It should be borne in mind that in the laboratory this is achievable with a unidirectional coupler, utilizing Faraday rotation, which involves the alignment of a magnetic field with a direction of propagation, effectively breaking reflection invariance of the optical system. We will find that we must do the same in our Hamiltonian.

We shall adapt the input-output methods of [13–15] and write the Hamiltonian for our system as

$$H = H_{\rm sys} + \int_{-\infty}^{\infty} d\omega\, \hbar|\omega| b^\dagger(\omega) b(\omega) + i\hbar \int_{-\infty}^{\infty} d\omega\, \kappa_1(\omega) \left\{\sigma_1^- b^\dagger(\omega) - \sigma_1^+ b(\omega)\right\}$$
$$+ i\hbar \int_{-\infty}^{\infty} d\omega\, \kappa_2(\omega) \left\{\sigma_2^- b^\dagger(\omega)\, e^{-i\omega\tau} - \sigma_2^+ b(\omega)\, e^{i\omega\tau}\right\}. \tag{1}$$

We interpret the $b(\omega)$ to be defined so that the (one dimensional) electric field is

$$A(x,t) = A^+(x,t) + A^-(x,t) \tag{2}$$

with

$$A^+(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega\, b(\omega,t)\, e^{i\omega x/c}. \tag{3}$$

Here $\sigma_1^\pm$, $\sigma_2^\pm$ are independent Pauli matrices for the different atoms (in fact, however, they can be interpreted as arbitrary operators if more general systems are being considered). For the case of two two-level atoms being considered here $H_{\rm sys}$ is defined as

$$H_{\rm sys} = \frac{1}{2}\hbar\Omega\sigma_1^z + \frac{1}{2}\hbar\Omega\sigma_2^z, \tag{4}$$

but it can also be interpreted more generally.

The quantity τ is such that $c\tau$ is the distance between the two atoms, and for definiteness will be considered positive in what follows. The reason for this interpretation will become clear shortly. Although we write everything in terms of ω, which can take on both positive and negative values, the integrals in (1) and (3) are really over a wave-number variable given by ω/c, so that ω is not really a frequency. The commutation relations are of course $[b(\omega), b^\dagger(\omega')] = \delta(\omega - \omega')$.

The equation of motion for $b(\omega)$ is

$$\dot{b}(\omega,t) = -i|\omega| b(\omega,t) + \kappa_1(\omega)\sigma_1^-(t) + \kappa_2(\omega)\sigma_2^-(t)\, e^{-i\omega\tau}, \tag{5}$$

which can be integrated to give

$$b(\omega,t) = e^{-i|\omega|(t-t_0)}b_0(\omega) + \int_{t_0}^{t} dt'\, e^{-i|\omega|(t-t')}\left\{\kappa_1(\omega)\sigma_1^-(t) + \kappa_2(\omega)\sigma_2^-(t)\, e^{-i\omega\tau}\right\} . \tag{6}$$

A Markov approximation can now be made, as in [15, 16]. We assume that all coupling is within a rather narrow bandwidth Δ of Ω and that

$$\kappa_1(\Omega) = \sqrt{\frac{\gamma_1}{2\pi}}, \quad \kappa_2(\Omega) = \sqrt{\frac{\gamma_2}{2\pi}},$$
$$\kappa_1(-\Omega) = \kappa_2(-\Omega) = 0 . \tag{7}$$

Thus, we break reflection invariance by making the κ values for positive and negative ω different from each other.

We then can write an expression for $A^+(x,t)$ as in (3), and make the approximation

$$\int_{\Omega-\Delta}^{\Omega+\Delta} d\omega\, e^{i\omega(t-t')}\kappa_1(\omega) \approx \sqrt{2\pi\gamma_1}\delta(t-t') . \tag{8}$$

This delta function will in fact be only an approximate delta function with a width in time $\sim \Delta^{-1}$, which will in practice be far smaller than any decay constants involved.

We finally arrive at the expression

$$\begin{aligned} A^+(x,t) = {} & b_{\rm in}(t-x/c) + b'_{\rm in}(t+x/c) \\ & + u(x/c)\sqrt{\gamma_1}\sigma_1^-(t-x/c) \\ & + u(x/c-\tau)\sqrt{\gamma_2}\sigma_2^-(t-x/c+\tau) , \end{aligned} \tag{9}$$

where

$$u(t) = \begin{cases} 1, & t > 0, \\ 0, & t < 0 \end{cases} \tag{10}$$

is a step function.

This represents a field with both left-to-right and right-to-left propagating parts. The right-to-left propagating part is a free field $b'_{\rm in}(t+x/c)$, which does not interact with either atom. The left-to-right propagating part is the remainder of the expression, which has an incoming free field, which interacts with the two atoms, at the points $x = 0$ and $x = c\tau$, respectively. The terms $\sqrt{\gamma_2}\sigma_1^-(t-x/c)$ and $\sqrt{\gamma_2}\sigma_2^-(t-x/c+\tau)$ represent radiation from the first and second atoms, respectively. The distance between the atoms is clearly $c\tau$, and the difference in the arguments of σ_1^-, σ_2^-, represents the time τ for light to pass from one atom to the other.

If now a is any operator in the Hilbert space of the two atoms we can write the equation of motion

$$\begin{aligned} \dot a = {} & -\frac{i}{\hbar}[a, H_{\rm sys}] \\ & + \int d\omega\kappa_1(\omega)\left\{[\sigma_1^-, a]b^\dagger(\omega,t) - b(\omega,t)[\sigma_1^+, a]\right\} \\ & + \int d\omega\kappa_2(\omega)\left\{[\sigma_2^-, a]b^\dagger(\omega,t) - b(\omega,t)[\sigma_2^+, a]\right\} . \end{aligned} \tag{11}$$

The ω integrals are actually restricted to the interval $(\Omega-\Delta, \Omega+\Delta)$, so that we can, using (7), express the integrals over $b(\omega,t)$, $b^\dagger(\omega,t)$ in terms of only the left-to-right propagating part. Using (7) and (9), we get

$$\begin{aligned} \dot a = {} & -\frac{i}{\hbar}[a, H_{\rm sys}] - [a,\sigma_1^+]\left\{\frac{\gamma_1}{2}\sigma_1^- + \sqrt{\gamma_1}b_{\rm in}(t)\right\} + \left\{\frac{\gamma_1}{2}\sigma_1^+ + \sqrt{\gamma_1}b_{\rm in}^\dagger(t)\right\}[a,\sigma_1^-] \\ & -[a,\sigma_2^+]\left\{\frac{\gamma_1}{2}\sigma_2^- + \sqrt{\gamma_1\gamma_2}\sigma_1^-(t-\tau) + \sqrt{\gamma_2}b_{\rm in}(t-\tau)\right\} \\ & + \left\{\frac{\gamma_2}{2}\sigma_2^+ + \sqrt{\gamma_1\gamma_2}\sigma_1^+(t-\tau) + \sqrt{\gamma_2}b_{\rm in}^\dagger(t-\tau)\right\}[a,\sigma_2^-] . \end{aligned} \tag{12}$$

This is our central equation. Notice that if a is an operator of the first atom, then we can set $[a,\sigma_2^+] = [a,\sigma_2^-] = 0$, and we obtain a quantum Langevin equation for atom 1 only: The first atom therefore does not feel any effect from the second atom. In contrast, the second atom *is* influenced by the output from the first atom.

This result can be seen to lead to a natural interpretation, since the equation of motion for a system operator of the second atom is just the usual quantum Langevin [13–15] equation, but with an input field equal to the output field from atom 1, delayed by τ.

It is clear that we can similarly find equations of motion for systems in which there are more than two atoms, and in which succeeding atoms in a chain are driven by the output of the previous atom. In the case that there is no feedback, i.e., no atom feeds its output into an atom which is previous to it in the chain, the delay τ is essentially an arbitrary constant, since all results for a given delay can be obtained from those with another value for τ by appropriate adjustments. We therefore let $\tau \to 0+$ in (12). From the resulting equation we can derive a master equation for the reduced two atom density operator in the Schrödinger picture $\rho = {\rm Tr}_{\rm field}\{\rho_{\rm tot}\}$ by computing

$${\rm Tr}\left\{\dot a \rho_{\rm tot}^H\right\} = {\rm Tr}_{\rm atoms}\left\{a_S\frac{\partial\rho}{\partial t}\right\} , \tag{13}$$

where $\rho_{\rm tot}^H$ is the Heisenberg picture density operator for the whole system. We consider in this paper only the case in which $b_{\rm in}(t)$ is a vacuum field, so that $\rho_{\rm tot}b_{\rm in}^\dagger(t) = b_{\rm in}(t)\rho_{\rm tot} = 0$. More general situations involve the use of quantum Ito calculus, and will be treated elsewhere.

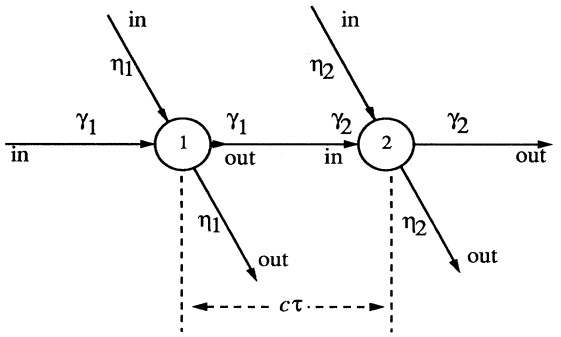

FIG. 1. Schematic diagram of the coupling of the two atoms to each other and to other modes.

In writing this master equation, we also generalize to the situation in which the atoms are coupled to other modes with coupling constants η_1, η_2, and these represent the fact that the driving fields are not perfectly coupled to the atoms and also allow us to couple in a driving field. Going into the interaction picture, we find the master equation for an incident coherent electric field E in the κ_1 channel,

$$\begin{aligned}\frac{\partial\rho}{\partial t} &= \frac{\gamma_1+\eta_1}{2}\left(2\sigma_1^-\rho\sigma_1^+ - \rho\sigma_1^+\sigma_1^- - \sigma_1^+\sigma_1^-\rho\right)\\ &+\frac{\gamma_2+\eta_2}{2}\left(2\sigma_2^-\rho\sigma_2^+ - \rho\sigma_2^+\sigma_2^- - \sigma_2^+\sigma_2^-\rho\right)\\ &-\sqrt{\gamma_1\gamma_2}[\sigma_2^+,\sigma_1^-\rho] + \sqrt{\gamma_1\gamma_2}[\rho\sigma_1^+,\sigma_2^-]\\ &-\sqrt{\eta_1}[E\sigma_1^+ - E^*\sigma_1^-,\rho]\,. \qquad (14)\end{aligned}$$

The situation is illustrated in Fig. 1. Notice that if we trace out over the second atoms we get a master equation for atom 1 in which there is no influence of the second atom, but we cannot do this for atom 2. Furthermore, if we use the same technique for a two-level atom driven by a degenerate parametric oscillator, and use a positive-P representation [10, 15, 17] for the oscillator variables, we get exactly the equations already used in [8–10] for the analysis of that problem.

The solution of (14) is quite straightforward. We compute the stationary normally ordered intensity correlation functions of the output fields, which are given by [1, 15]

$$\langle I_i(t)I_j(0)\rangle_s = \gamma_i\gamma_j \mathrm{Tr}\{\sigma_i^+\sigma_i^- V(t)\left\{\sigma_j^-\rho_s\sigma_j^+\right\}\}\,, \qquad (15)$$

where $V(t)$ is the (two-sided) evolution operator, such that $V(t)\{\rho(t')\} = \rho(t+t')$, and ρ_s is the stationary solution of the master equation. The correlation functions are related to photon counting, and can be measured by the methods of [18]. They represent essentially the conditional probability of counting a photon from atom i a time t after counting a photon from atom j.

This kind of equation is already too complicated for any analytical results to be of any use. A numerical solution of the fifteen coupled differential equations (12) is readily obtained giving solutions for $V(t)$, ρ_s, and hence $\langle I_i(t)I_j(0)\rangle_s$, in a matter of seconds on any modern workstation. A typical set of results of these are shown in Fig. 2. Features which appear for all reasonable ranges of parameters are the following

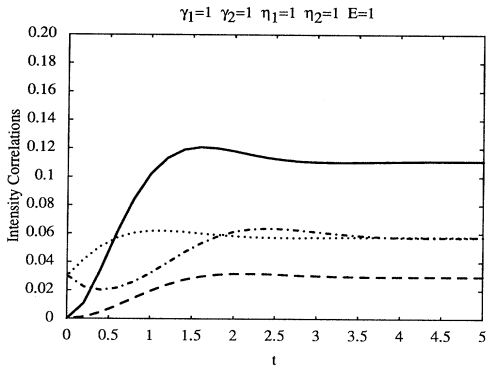

FIG. 2. Plots of (i) solid line, $\langle I_1(t)I_1(0)\rangle_s$; (ii) dashed line, $\langle I_2(t)I_2(0)\rangle_s$; (iii) dotted line, $\langle I_1(t)I_2(0)\rangle_s$; (iv) dot-dashed line, $\langle I_2(t)I_1(0)\rangle_s$.

(i) A slight enhancing of the antibunching near $t = 0$ in the light from second atom compared with that of the first atom.

(ii) A pronounced anticorrelation in $\langle I_2(0)I_1(0)\rangle_s$. This arises because of the antibunching. If a photon is measured in beam 1, then it is not available to excite atom 2. The anticorrelation is not perfect, since emission depends on the excitation of each atom—however, because of the antibunching, if a photon was counted in beam 1, then there was a reduced probability of there having been one previously to excite atom 2.

(iii) This anticorrelation becomes more pronounced in $\langle I_2(t)I_1(0)\rangle_s$, which initially decreases as t increases. This is because there will be no further photons to excite atom 2 in the time immediately following the emission. Eventually, of course, photons do appear, and $\langle I_2(t)I_1(0)\rangle_s$ rises to a value representing zero correlation.

(iv) However, this effect is not present in $\langle I_1(t)I_2(0)\rangle$, and this correlation simply rises to reach its equilibrium value.

The formalism presented here is quite general, and provides the basis for a "modular" or "network" quantum optics, completing the idea originated by Yurke and Denker [19]. This is a necessary tool if communication systems based on nonclassical states of light are to be realized. A more detailed development will be published elsewhere.

Carmichael [20] has independently developed a similar formalism which also appears in this issue, and I wish to thank him for agreeing to this method of publication. I wish to thank JILA for hospitality where most of this

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Note added.—On completion of this work it was brought to my attention that work by Kolobov and Sokolov [21] has developed similar equations in a different context.

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