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Consistent Interpretation of Quantum Mechanics Using Quantum Trajectories

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The probabilistic element of quantum theory can be combined with the unitary time evolution of Schrödinger's equation in a natural and consistent way using the idea of a quantum trajectory, the quantum analog of the trajectory traced out in phase space as a function of time by a point representing the state of a closed classical system. A family of quantum trajectories can be defined using bases for the quantum Hilbert space at different times chosen so that an appropriate noninterference condition, related to the Gell-Mann and Hartle notion of medium decoherence, is satisfied. The result is a generalization of the consistent histories approach to quantum mechanics.

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Ever since its inception, quantum mechanics has suffered from a severe conceptual difficulty in that it combines a unitary (thus "deterministic") time evolution, provided by Schrödinger's equation, with a stochastic or probabilistic element first introduced by Born [1,2]. While in practice both must be employed if quantum mechanics is to be applied to laboratory experiments, attempts to put the two together in a fully consistent way have encountered numerous difficulties. Thus if there are "hidden variables" which underlie the stochastic behavior of quantum systems, it seems they must be connected with a mysterious "action at a distance" difficult to reconcile with relativity theory [3]. The unitary time evolution, on the other hand, can result in coherent superpositions of macroscopically distinct states (the Schrödinger's cat paradox) which are difficult to interpret [4].

The notion of a *quantum trajectory*, developed in the present paper through analogy with the path traced out in time in a classical phase space by the point representing a closed classical system, seems capable of uniting the deterministic and stochastic elements of quantum theory in a natural way, using only the standard tools of nonrelativistic quantum theory. In particular, there are no hidden variables, and the usual unitary time evolution implied by Schrödinger's equation is employed. The resulting structure has no mysterious "action at a distance" of the type supposedly needed to accommodate violations of Bell's inequality [3]. The idea of a quantum trajectory is closely related, both in general spirit and in some technical details, to the "consistent history" approach to quantum mechanics [5–10]. See the comments at the end of the paper.

The mechanical properties of a closed classical system can be represented at any time by means of a point γ in the classical *phase space* Γ whose coordinates are the positions and momenta of the particles which constitute the system. In the course of time this point traces out a trajectory, determined by Hamilton's equations. Various physical properties ("the energy is between E_0 and E_1"; "there are five particles in the left side of the box"), which I shall call *events*, are represented by cells consisting of all points in Γ for which the property in question is true. Let cell C_E correspond to the event E. Then the orthogonal complement in Γ of the cell C_E, consisting of all points in Γ which are not in C_E, is the cell corresponding to the event "not E." A *history* is a sequence of events at different times; it occurs if the trajectory passes through the corresponding cells at the appropriate times, and otherwise it does not occur (for this trajectory). In classical statistical mechanics one assigns probabilities to the trajectories, and the probability that a history occurs is then the sum of the probabilities of the trajectories for which it occurs.

In a closed quantum system the Hilbert space $\mathcal{H}$ is the analog of the classical phase space Γ, and a ray, or one-dimensional subspace, or wave function is the counterpart of the point γ. To simplify the exposition we shall assume $\mathcal{H}$ is of finite dimension. Quantum events correspond to linear subspaces of $\mathcal{H}$; e.g., the event that the energy is between E_0 and E_1 corresponds to the subspace spanned by all eigenvectors of the Hamiltonian having eigenvalues in this range. If the event E corresponds to the subspace S_E, the event "not E" corresponds to its orthogonal complement $S_E^{\perp}$, the subspace consisting of all vectors in $\mathcal{H}$ orthogonal to every vector in S_E. The quantum Hamiltonian determines the time evolution through the unitary time transformation

$$U(t_2 - t_1) = e^{-i(t_2-t_1)H/\hbar}. \quad (1)$$

It is rather natural to suppose that the quantum counterpart of a classical trajectory is the time-dependent wave function

$$|\psi(t)\rangle = U(t)|\psi(0)\rangle \quad (2)$$

starting from the initial state $|\psi(0)\rangle$, and that a quantum history, consisting of events at successive times, actually occurs if $|\psi(t)\rangle$ lies in the appropriate subspace of $\mathcal{H}$ at each time in question. However, there are difficulties in supposing that quantum trajectories must be of the form (2). First, it is often the case that for events which interest us, such as the position of the pointer of a macroscopic apparatus after a measurement, $|\psi(t)\rangle$ lies neither in the subspace corresponding to the event nor in its orthogonal complement, so that we cannot say whether or not the event occurs. Second, there is no obvious way of introducing a stochastic element into the discussion if one assumes the initial state is a pure state, corresponding to a specific initial wave function.

Both problems are solved simultaneously by introducing a more general definition of a *quantum trajectory*. Consider a set of times $t_1 < t_2 < \cdots < t_n$, and for each t_j choose an orthonormal basis $\{|\phi_j^\alpha\rangle\}$ of the Hilbert space, where α indexes the basis vectors. Construct the *trajectory graph*, Fig. 1, in which all the basis vectors at a particular time are represented by nodes placed in a vertical column, and lines are drawn between the nodes (j,α) and $(j+1,\alpha')$ if and only if

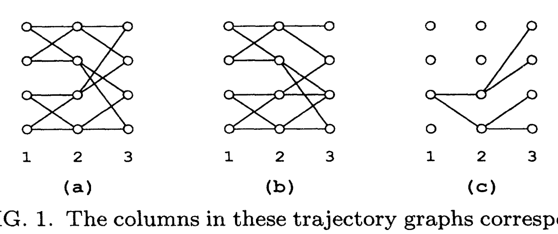

FIG. 1. The columns in these trajectory graphs correspond to three times $t_1 < t_2 < t_3$. The noninterference condition is satisfied for (a) but not for (b): note the two paths connecting the next-to-lowest nodes at t_1 and t_3 in the latter. An elementary family of those trajectories in (a) passing through one of the nodes at time t_1 in (a) is shown in (c).

$$\langle\phi_{j+1}^{\alpha'}|U(t_{j+1} - t_j)|\phi_j^\alpha\rangle \quad (3)$$

is nonzero. A *path* on this graph is defined as a succession of consecutive nodes, $(j,\alpha_j), (j+1,\alpha_{j+1}), (j+2,\alpha_{j+2}), \ldots, (j+k,\alpha_{j+k})$ connected by lines, or the same sequence in reverse order. (Note that we do not consider paths which correspond to a nonmonotonic sequence of times.) Provided any pair of nodes at different times are connected by *at most one path*, we shall say that the graph, or equivalently the choice of bases, satisfies the *noninterference condition*, and in this case we shall refer to the individual paths extending from t_1 to t_n as *trajectories*. In addition to the *complete family* of all trajectories, we shall be interested in the *elementary family* consisting of all the trajectories which pass through a particular node.

A single quantum trajectory may be regarded as a generalization to a sequence of times of the notion of a pure quantum state at a single time: it constitutes the most precise description which quantum mechanics can provide about the state of the system at the times in question. However, not every sequence of pure states connected by nonvanishing matrix elements (3) constitutes a trajectory, for as soon as three or more times are involved, the requirement that a trajectory be part of a complete family satisfying the noninterference condition is nontrivial. Fortunately, in many situations of interest there are ways of checking the noninterference condition without explicitly constructing the complete family; for example, see the remarks following Eq. (4) below.

Each quantum trajectory is assigned a *weight* equal to the product of the absolute squares of the matrix elements (3) associated with the lines forming the corresponding path, and these weights can be used to generate a probability distribution on a family of trajectories in the following way. First, consider the elementary family associated with some node; this node can (but need not) be an "initial state" at the time t_1, as in Fig. 1(c). The probability of each trajectory passing through this node is then equal to its weight (it is straightforward to show that the sum of these weights is 1), and all other trajectories have zero probability. Next, one can construct a quantum statistical mechanics, analogous to its classical counterpart, by choosing a particular time t_j and assigning some probability distribution to the various nodes $\{|\phi_j^\alpha\rangle\}$ at this time. The probability assigned to each node is then divided up among the trajectories passing through it in proportion to the weight of each trajectory. In all cases it is the trajectories which form the elementary objects in the probabilistic sample space, and the probability of any node is, by definition, the sum of the probabilities of the trajectories which pass through it.

The noninterference condition plays a crucial role in ensuring that the probabilities defined in this way make physical sense. Thus, for example, the probability that the system is in a particular final state at time t_n, given that it was in a specified initial state at time t_1, should not depend on how many intermediate times are considered, or the choice of the orthonormal bases at these times. In particular, if the initial state is $|\psi(0)\rangle$ and the final state is orthogonal to $|\psi(t_n)\rangle$ as defined by (2), this probability should be zero. It is the noninterference condition which ensures that this and analogous requirements of physical consistency are satisfied. (See the end of the paper for some additional remarks.)

Given a probability distribution for trajectories, one can discuss the probability of occurrence of a history consisting of events $E_1, E_2, \ldots, E_n$ at the times $t_1, t_2, \ldots, t_n$, provided this history is *compatible* with the probability distribution in the sense that for every time t_j and every node $|\phi_j^\alpha\rangle$ *having nonzero probability*, either $|\phi_j^\alpha\rangle$ falls in the subspace S_j of $\mathcal{H}$ corresponding to E_j ("E_j occurs") or in its orthogonal complement $S_j^\perp$ ("E_j does not occur"). For a given trajectory having nonzero probability, the (compatible) history occurs if and only if for each j the $|\phi_j^\alpha\rangle$ for this trajectory falls in S_j. The probability of the history is then the sum of the probabilities of the trajectories for which it occurs, just as in the case of classical statistical mechanics.

In many applications one is not interested in a complete family of trajectories, and the following construction is a useful one. Let $S_1, S_2, \ldots, S_n$ be subspaces of $\mathcal{H}$ which are nested in the sense that

$$U(t_{j+1} - t_j)S_j \subset S_{j+1}, \tag{4}$$

let $\{|\phi_j^\alpha\rangle\}$ be an orthonormal basis of S_j, and suppose that the trajectory graph constructed using these $\{|\phi_j^\alpha\rangle\}$ satisfies the noninterference condition. [In particular, if S_1 is one dimensional, corresponding to a pure initial state, all the trajectories emanate from this initial state, a single node, at time t_1 and do not intersect at any later time, so they constitute an elementary family associated with the initial state; see Fig. 1(c).] Then one can show that the corresponding family can always be embedded in a complete family, i.e., the trajectory graph is a part of a larger graph corresponding to a complete basis of $\mathcal{H}$ for each t_j, $1 \leq j \leq n$, which satisfies the noninterference condition. The same construction will also work if "$\subset$" in (4) is replaced by "$\supset$," since, obviously, no direction (or sense) of time is singled out when defining a complete family.

It must be stressed that there are a large number of different possibilities for constructing families of trajectories; and different families are typically incompatible with one another and cannot be combined, because this will violate the noninterference conditions. Probabilities of histories based on distinct incompatible families of trajectories cannot be compared, because they do not belong to a single probability structure. The crucial importance of not comparing incompatible or complementary families of consistent histories has been stressed by Omnès [11], and the same considerations apply to quantum trajectories. This crucial difference between the quantum and the classical world is illustrated by the following example, which is based on the construction of the preceding paragraph using a pure initial state.

Consider (Fig. 2) a particle (neutron or photon) which passes through a beam splitter BS_1 into an interferometer, and then through a second beam splitter BS_2, which may be present or absent, to a pair of counters. If BS_2 is present, the particle arrives with probability 1 at counter C_A, and if BS_2 is absent, the particle will arrive with probability 1/2 at either counter. As we are discussing a closed system, the counters must be described quantum mechanically. Thus if $\epsilon C_A C_B$ represents the initial state, ϵ referring to the particle, the time evolution corresponding to (2) with the second beam splitter *absent* can be represented schematically, for $t_1 < t_2 < t_3$, by

$$\begin{aligned} \epsilon C_A C_B &\rightarrow (\alpha+\beta) C_A C_B/\sqrt{2} \\ &\rightarrow (C_A^t C_B + C_A C_B^t)/\sqrt{2}, \end{aligned} \tag{5}$$

where the intermediate state at a time t_2 corresponds to the particle in the interferomenter (α and β corresponding to the upper and lower arms, respectively), whereas the state at time t_3 arises from $\alpha C_A \rightarrow C_A^t$, $\beta C_B \rightarrow C_B^t$, the superscript t indicating the counter has been triggered.

Now (5) is a perfectly acceptable trajectory. The final state is a coherent superposition of two macroscopically distinct situations, and hence, using this trajectory, one can say nothing about whether counter A triggers or counter B triggers. By contrast, using the same initial state, but an alternative set of basis states at times t_2 and t_3, one obtains two trajectories with a common origin at time t_1,

$$\epsilon C_A C_B \rightarrow \begin{cases} \alpha C_A C_B \rightarrow C_A^t C_B, \\ \beta C_A C_B \rightarrow C_A C_B^t, \end{cases} \tag{6}$$

each of which has a weight, and thus a probability of 1/2. Thus there is a history in which the particle follows the α arm and triggers counter A, and another history in which it follows the β arm and triggers counter B. These two histories form a consistent family, and each occurs with probability 1/2. On the other hand, neither is compati-

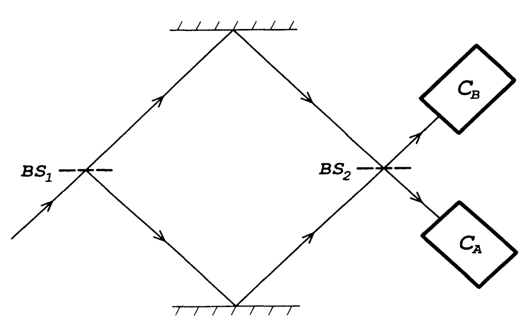

FIG. 2. Interferometer with two beam splitters, BS_1 and BS_2, and two counters, C_A and C_B.

ble with the trajectory (5), and hence it makes no sense to ask whether one of them occurred "instead of" some history based upon (5). Quantum mechanics allows us to tell a variety of different stories about a system, but a question of the form "Which of these *really* took place?" asked in terms of comparing two mutually incompatible histories, makes no sense quantum mechanically.

If the second beam splitter is inserted, the trajectory corresponding to (2) is now

$$\epsilon C_A C_B \to (\alpha+\beta) C_A C_B/\sqrt{2} \to C_A^t C_B. \tag{7}$$

Once again, this is a perfectly acceptable trajectory, and it implies that the history which ascribes a superposition state to the particle at t_2, and that C_A (and not C_B) has been triggered by the particle at time t_3, occurs with probability 1. On the other hand the two trajectories

$$\epsilon C_A C_B \to \begin{cases} \alpha C_A C_B \to (C_A^t C_B + C_A C_B^t)/\sqrt{2}, \\ \beta C_A C_B \to (C_A^t C_B - C_A C_B^t)/\sqrt{2}, \end{cases} \tag{8}$$

also form an elementary family, and each can be assigned probability 1/2. Now while there are a variety of reasons why the practical minded physicist will usually prefer to discuss (7) rather than (8), which corresponds to histories which have no very obvious laboratory interpretation, there is no reason in principle why (8) must be excluded from consideration, any more than (6). Quantum mechanics does not tell us that (7) must occur rather than one of the possibilities in (8); it simply tells us that asking whether one occurs rather than the other makes no sense.

The relation of quantum trajectories to the consistent histories approach to quantum mechanics arise through the fact that the noninterference condition following (3) is, in effect, a type of consistency condition. Unlike the consistency conditions of Omnès [7,8] and the decoherence conditions of Gell-Mann and Hartle [9,10], the noninterference condition is manifestly time symmetric: it is unchanged if the time axis is reversed. But it also differs from the time-symmetric condition of Griffiths [5] in that the initial and final times play no distinguished role. By means of the result stated below (4), one can show that the "generalized records" which Gell-Mann and Hartle [10] obtain by applying their "medium decoherence condition" to a *pure* initial state are the same as the elementary family of quantum trajectories associated with this state, given a suitable choice of the bases at later times. In the case of a density matrix at the initial time, the noninterference condition is a particular instance of what Gell-Mann and Hartle call the "medium-strong decoherence condition," when that is appropriately interpreted. The noninterference condition is definitely stronger than the consistency conditions employed by Griffiths and by Omnès. However, many of the specific examples considered by these authors (see, e.g., the discussion of the EPR paradox in [6]) satisfy the stronger condition, and thus can be discussed equally well using quantum trajectories, with identical results. It is also possible that there is a satisfactory alternative definition of quantum trajectories employing a noninterference condition which is weaker than that used in this paper.

The line of thought presented here began during a workshop held at the Aspen Center for Physics during June of 1992. I am indebted to several of the participants, in particular M. Gell-Mann, J. Hartle, R. Omnès, W. G. Unruh, and W. Zurek, for some very lively and useful discussions. Financial support for this research has come from the National Science Foundation through Grant No. DMR-9009474.

[1] A convenient summary of the history of quantum mechanics will be found in M. Jammer, *The Philosophy of Quantum Mechanics* (Wiley, New York, 1974).

[2] A very useful collection of reprints or translations of original papers will be found in *Quantum Theory and Measurement*, edited by J. A. Wheeler and W. H. Zurek (Princeton Univ. Press, Princeton, 1983).

[3] There is an enormous literature on this subject. Quite readable discussions will be found in: N. D. Mermin, Am. J. Phys. **49**, 940 (1981); **58**, 731 (1990); F. Selleri, *Quantum Paradoxes and Physical Reality* (Kluwer Academic, Dordrecht, 1990), among others.

[4] See E. P. Wigner, Am. J. Phys. **31**, 6 (1963), reprinted in [2], for a very clear presentation.

[5] R. B. Griffiths, J. Stat. Phys. **36**, 219 (1984); in *Fundamental Questions in Quantum Mechanics*, edited by L. M. Roth and A. Inomata (Gordon and Breach, New York, 1986), p. 211; in *New Techniques and Ideas in Quantum Measurement Theory*, edited by D. M. Greenberger (New York Academy of Sciences, New York, 1986), p. 512.

[6] R. B. Griffiths, Am. J. Phys. **55**, 11 (1987).

[7] R. Omnès, J. Stat. Phys. **53**, 893 (1988), and items cited in [8].

[8] R. Omnès, Rev. Mod. Phys. **64**, 339 (1992).

[9] M. Gell-Mann and J. B. Hartle, in *Complexity, Entropy, and the Physics of Information*, edited by W. Zurek (Addison-Wesley, Reading, 1990), and items cited in [8].

[10] M. Gell-Mann and J. B. Hartle, in *Proceedings of the Twenty-Fifth International Conference on High Energy Physics, Singapore, 1990*, edited by K. K. Phua and Y. Yamaguchi (World Scientific, Singapore, 1990); (to be published).

[11] See [8], Sec. II.C.