Nonlinear Wave Evolution in the Expanding Solar Wind

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We report here on a numerical model allowing direct numerical simulations of magnetohydrodynamic fluctuations advected by the expanding solar wind. We show that the expansion of the plasma delays and possibly freezes the turbulent evolution, but that it also triggers the nonlinear evolution of otherwise stable (Alfvén) waves, which can thus release their energy in the wind.

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The sun is the source of a radially expanding supersonic flow, the solar wind, which reaches its cruise speed after some tens of solar radii. The dynamic evolution of imbedded fluctuations has been generally considered from two independent and apparently opposite viewpoints: nonlinear (or turbulent) and linear (wave propagation in an expanding medium). However, both kinds of interaction have to be considered together if we are to understand the main observational and theoretical problem which justifies much of the interest for solar wind turbulence: how and when the fluctuations release their energy in the (expanding) plasma during transport. We propose in this Letter a framework to understand both nonlinear and linear interactions and consider the specific example of finite amplitude plane waves; the results obtained indicate that the dynamics of MHD turbulence is drastically modified by the expansion of the plasma.

The question of turbulent dissipation in the solar wind may be described as follows. On the one hand, the cooling of the plasma with distance, which basically follows the adiabatic prediction for a radial expansion with constant speed (temperature $\propto R^{-4/3}$) is observed to be slower in fast streams, where the wave flux is important [1]: this seems to indicate "turbulent" heating. On the other hand, in the frequency range where nonlinear transfer could have time to take place, fluctuations ("Alfvénic waves") resemble freely propagating waves, i.e., nonlinear interactions seem to be much reduced [2,3]. Nonetheless, the spectra of the fluctuations evolve with distance, a clear indication that nonlinear interactions are at work [2]. To understand how a nonlinear cascade may take place, we solve the full nonlinear magnetohydrodynamic equations in an expanding mean flow. This may be viewed as a continuation of the early work by Tu, Pu, and Wei [4], who first studied this problem using the simplifying assumptions of small scale incompressibility, isotropy, and a dimensional estimate of nonlinear interactions.

Simulating turbulent dissipation within, say, the inner heliosphere using an absolute frame of reference is costly in terms of computer memory and time. We bypass this difficulty by using a coordinate system moving with the average flow speed: the whole computer memory is hence available for small-scale (turbulent) dynamics, as in standard homogeneous simulations. A first possibility is then to use a spherical coordinate system r,θ,ϕ centered on the sun and assuming a constant radial mean velocity: $\mathbf{U}=U^0\hat{\mathbf{e}}_r$; adopting the comoving radial coordinate $r'=r-R(t)$, where $R(t)=R^0+U^0t$ is the average Lagrangian heliocentric distance at time t, the MHD equations may be integrated within $|r'|\leq L^0/2, |\phi|\leq \alpha/2, |\theta| \leq \alpha/2$. We prefer, however, to use the simpler Cartesian coordinates; this is acceptable provided the box is small enough, i.e., $\alpha \ll 1$ and $L^0/R \ll 1$. Consider a Cartesian frame with x axis parallel to the radial passing through the middle of the box, change to the Galilean frame moving with the mean wind along this radial $[x'=x-U^0t =x-R(t)]$. In the new frame, the mean velocity has a residual x component $O(\alpha^2)$ which we neglect, and a transverse $[O(\alpha)]$ component $\mathbf{U}_\perp$ with maximum value $\delta U=\alpha U^0/2$ which we retain, so that the mean velocity

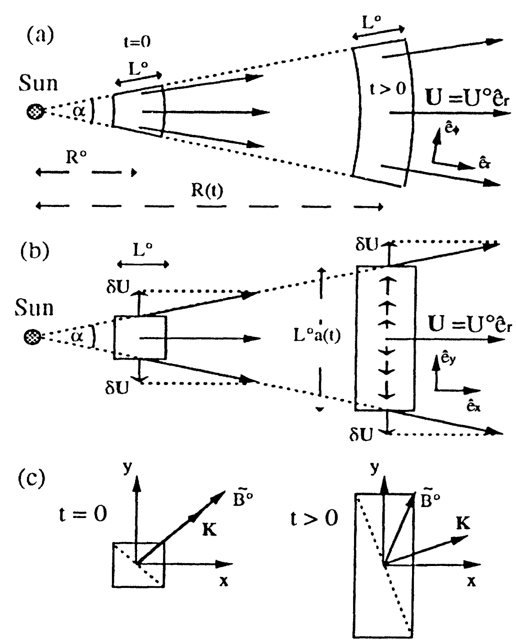

FIG. 1. Sketch of the evolution of a plasma volume advected by a spherical wind with constant speed. (a) Exact evolution, (b) approximate evolution in the limit of small angular size, and (c) transformation of a parallel wave ($\mathbf{k}\parallel\mathbf{B}^0$) into an oblique wave.

reduces to

$$\mathbf{U}_\perp = \delta U\{(2y/L^0 a)\hat{\mathbf{e}}_y + (2z/L^0 a)\hat{\mathbf{e}}_z\} = \dot{a}/a(y\hat{\mathbf{e}}_y + z\hat{\mathbf{e}}_z).$$

$L^0 a$ is the transverse size of the box, which grows linearly with time, a being the aspect ratio $a(t) = R(t)/R^0 = 1 + Ut/R^0$, where R^0 is the initial heliocentric distance. Within this approximation, an initially cubic box is uniformly stretched in the transverse directions and becomes a parallelepiped [Figs. 1(a) and 1(b)]. By moving to expanding "comobile" coordinates $\tilde{t} = t$, $\tilde{x} = x'$, $\tilde{y} = y/a(t)$, and $\tilde{z} = z/a(t)$, we eliminate the transverse advection terms of the form $\mathbf{U}_\perp \cdot \nabla$. This has the advantage of allowing periodicity of all fields, density ρ, pressure P, magnetic field $\mathbf{B}$, and velocity fluctuation $\mathbf{u}$, in the transverse comobile coordinates; since the thickness of the box is small, we also assume periodicity in the radial (x) direction. Omitting dissipation terms, the equations take the form of standard MHD equations, with additional linear terms involving the mean velocity $\mathbf{U}_\perp$ appearing in the right-hand side:

$$\begin{aligned}
&\partial \mathbf{u}/\partial \tilde{t} + \mathbf{u}\cdot\nabla\mathbf{u} - \mathbf{B}\cdot\nabla\mathbf{B}/\rho + \nabla(P + B^2/2)/\rho = -(\mathbf{u}\cdot\nabla)\mathbf{U}_\perp = -\dot{a}/a\mathbb{T}\cdot\mathbf{u}\,,\\
&\partial \mathbf{B}/\partial \tilde{t} + \mathbf{u}\cdot\nabla\mathbf{B} - \mathbf{B}\cdot\nabla\mathbf{u} + \mathbf{B}\,\mathrm{div}\mathbf{u} = -\mathbf{B}\,\mathrm{div}\mathbf{U}_\perp + (\mathbf{B}\cdot\nabla)\mathbf{U}_\perp = -\dot{a}/a\mathbb{L}\cdot\mathbf{B}\,,\\
&\partial P/\partial \tilde{t} + \mathbf{u}\cdot\nabla P + \gamma P\,\mathrm{div}\mathbf{u} = -\gamma P\,\mathrm{div}\mathbf{U}_\perp = -2\gamma\dot{a}/aP\,,\\
&\partial \rho/\partial \tilde{t} + \mathrm{div}(\rho\mathbf{u}) = -\rho\,\mathrm{div}\mathbf{U}_\perp = -2\dot{a}/a\rho\,;\\
&\nabla = [\partial/\partial\tilde{x}, (1/a)\partial/\partial\tilde{y}, (1/a)\partial/\partial\tilde{z}], \quad |\tilde{x}|,|\tilde{y}|,|\tilde{z}| \le L^0/2\,.
\end{aligned} \tag{1}$$

The last line gives the expression of the derivative operators in terms of comobile coordinates, and $\gamma = 5/3$ is the polytropic index. The matrices $\mathbb{T}$ and $\mathbb{L}$ are defined as $T_{ij} = T^i\delta_{ij}, L_{ij} = L^i\delta_{ij}$, with $T^i = (0,1,1), L^i = (2,1,1)$; they represent an anisotropic *friction*, which is in the transverse direction for the velocity fluctuation and in the radial direction for the reduced magnetic field $\tilde{\mathbf{B}} = \mathbf{B}/\sqrt{\langle\rho\rangle}$, where the average density falls as $\langle\rho\rangle = \rho^0/a^2(t) \propto R^{-2}$ [which corresponds to the conservation of mass, since the box volume grows as $a^2(t)$]. The magnetic flux is conserved during the expansion, since $\langle B\rangle_x \propto 1/a^2$, $\langle B\rangle_y \propto \langle B\rangle_z \propto 1/a$: the mean field thus rotates in the x-y plane (cf. Parker's spiral [5]). In the absence of dissipation, the average temperature $(T = P/\rho)$ falls as $\langle T\rangle \propto a^{-2(\gamma-1)} \propto R^{-4/3}$, as expected. For small amplitude Alfvén waves, i.e., transverse velocity and reduced magnetic fluctuations $\tilde{\mathbf{b}} = \tilde{\mathbf{B}} - \tilde{\mathbf{B}}^0$ (with constant pressure and density) superposed on a mean radial magnetic field $\tilde{\mathbf{B}}^0 = [\tilde{B}^0(t), 0, 0]$, Eqs. (1) become

$$\begin{aligned}
&\partial u/\partial t - (\tilde{B}^0/a)\partial\tilde{b}/\partial\tilde{x} = -\dot{a}/au\,,\\
&\partial\tilde{b}/\partial t - (\tilde{B}^0/a)\partial u/\partial\tilde{x} = 0\,.
\end{aligned} \tag{2}$$

In the limit of high frequencies, the eigensolutions of (2) are the "homogeneous" solutions $z^+ = 0$ and $z^- = 0$ (where $z^\pm = u \pm \tilde{b}$), which propagate in opposite directions along $\mathbf{B}^0$. When only one species (say z^+) is present, Eq. (2) implies that the energy decays as in the WKB approximation: $z^{+2} \approx u^2 \approx \tilde{b}^2 \approx 1/a \propto 1/R(t)$. Hence Eqs. (1) contain all the known linear scaling laws with heliocentric distance.

In the general (nonlinear) case, the expansion-induced friction delays nonlinear interactions, by reducing the fluctuation amplitude δu, since the typical time scale is of the order $\tau_{\mathrm{NL}} \equiv L^0/(2\pi\delta u)$, where L^0 is the typical wavelength. Moreover, the expansion causes an increase of the transverse wavelength, and so leads to an additional slowdown. We obtain an exact result concerning the slowdown in the extreme case of purely transverse velocity fluctuations $u(y)$, with $B = 0, P = 0$, which are described by (ν being the viscosity)

$$\partial u/\partial t + (1/a)u\partial u/\partial\tilde{y} = \nu(1/a^2)\partial^2 u/\partial\tilde{y}^2 - (\dot{a}/a)u\,, \quad |\tilde{y}| \le L^0/2\,. \tag{3}$$

We take δu as unit velocity, and the nonlinear time as unit time; the aspect ratio is $a(t) = 1 + \varepsilon t$, where $\varepsilon = \tau_{\mathrm{NL}}U/R^0 \approx \delta U/(\pi\delta u)$ is the parameter measuring the relative importance of expansion and nonlinear terms [4]. In the homogeneous case $\varepsilon = 0$, Eq. (3) reduces to Burgers equation: an initial smooth velocity profile steepens and forms a shock with thickness proportional to ν. In the limit of vanishing viscosity ν, the energy remains approximately constant before a shock forms, and subsequently decays as $\partial u^2/\partial t \approx -u^3/L^0$ so that $u^2 \approx 1/t^2$. The change of variables $w(\tilde{y},\theta) = au(\tilde{y},t)$, with $\theta = t/a(t)$, reduces Eq. (3) to the homogeneous Burgers equation: $\partial w/\partial\theta + w\partial w/\partial\tilde{y} = \nu\partial^2 w/\partial\tilde{y}^2$. At short times, $\theta \approx t$, $w \approx \mathrm{const}$ and the only evolution is the WKB decrease, here $u^2 \approx 1/R^2$; later on, the velocity profile steepens, but shock formation never occurs if the stretching factor ε is larger than unity. In that case, the energy decays uniformly as $1/R^2$. If $\varepsilon < 1$, the energy decays as $u^2 \approx 1/(t^2\theta^2) \approx 1/R^4$ for $1 < t < 1/\varepsilon$. The important point is that since $u(\tilde{y},t) = (1/a)w(\tilde{y},\theta)$ and $\theta(t) \to 1/\varepsilon$ when $t \to \infty$, the asymptotic wave form is given by the homogeneous solution w *frozen* at the finite time $1/\varepsilon$, and decays self-similarly as $1/a(t)$.

This conclusion may not apply strictly to more general situations, where the wave vectors have a nonzero component along the radial direction, which suffers no stretching: The wave vector will then turn towards the radial [6] [Fig. 1(c)], which is likely to decrease the efficiency of the freezing effect. The case of Alfvén-like

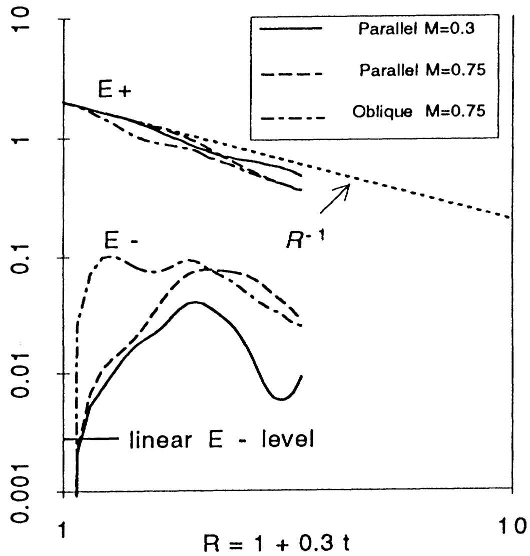

FIG. 2. Evolution of the energy content in circularly polarized Alfvén waves vs heliocentric distance R (arbitrary units) in the expanding wind.

fluctuations, which are often observed to dominate in the solar wind, provides an interesting example. Large amplitude circularly polarized Alfvén waves (constant density, magnetic and thermal pressures; $\mathbf{u}=\tilde{\mathbf{b}}, \mathbf{z}^-=0$) propagating along the magnetic field are exact solutions of the homogeneous MHD equations. However, in a radial wind, this holds *only* in the singular case of radial magnetic field and wave vector. Indeed, although there appears at finite frequency a nonzero z^- component of order $z^-/z^+ \approx \mathrm{div}U/8k\tilde{B}^0 \approx (\varepsilon/4)b^{\mathrm{rms}}/B^0$ [7,8], the coupling between z^- and z^+ remains zero. When $\mathbf{k}$ is not radial, the initially parallel wave ($\mathbf{k}\parallel\mathbf{B}^0$) transforms into an oblique wave, because the mean field B^0 turns away from the radial, while the wave vector on the contrary tends to align with the radial. In such an oblique wave, the pressure is no longer constant, so that the Alfvén wave steepens and a (one-dimensional) nonlinear cascade occurs.

We study the nonlinear evolution of a monochromatic circularly polarized Alfvén wave ($\mathbf{b}=\mathbf{u}$) in two versions: (a) a *parallel* case [$\mathbf{k}\parallel\mathbf{B}^0$, with $(\mathbf{k},\hat{\mathbf{e}}_r)=45^0$], and (b) an oblique case, for which $\mathbf{k}$ and $\mathbf{B}^0$ are not aligned from the start [we will take for simplicity a radial $\mathbf{B}^0$, $(\mathbf{k},\hat{\mathbf{e}}_r)=45^0$]. In case (b), the magnetic pressure is initially modulated, which triggers a nonlinear evolution even with $\varepsilon=0$ [9]. We take an Alfvén Mach number $M_A=b^{\mathrm{rms}}/B^0=0.5$, and $\varepsilon=\tau_{\mathrm{NL}}U/R^0=0.3$, and Mach numbers $M=u^{\mathrm{rms}}/c=0.3$ and 0.75 (c being sound speed) which corresponds respectively to a low β ($=0.25$) and high β ($=1.7$) plasma (the first case being more common in the solar wind [3]). The evolution is followed for 8 nonlinear times, the heliocentric distance $R(t)=R^0(1+\varepsilon t)$ thus increases by a factor of 3 (say from 0.3 up to 1 AU). Time

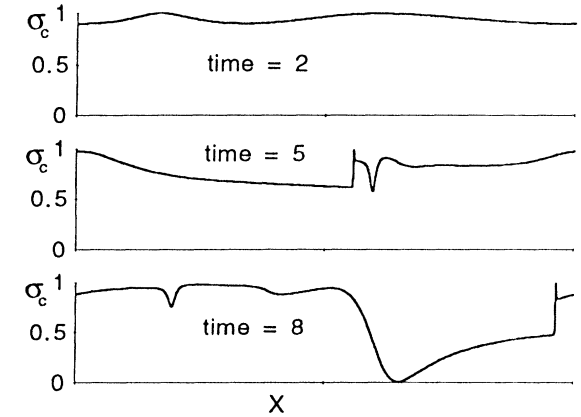

FIG. 3. Spatial profile of σ_c at several times (parallel case, $M=0.75$, $M_A=0.5$, $\varepsilon=0.3$); X is the coordinate along the propagation axis.

is advanced via the Adams-Bashforth method, a pseudospectral method is used. Small scale dissipation is achieved by standard molecular viscous terms: The large scale viscous time is about 250 nonlinear times, i.e., dissipation is negligible at energy containing scales (as required to compute turbulent dissipation), resolution is $N=1024$ grid points.

Figure 2 shows the evolution of "energies" $E^{\pm}=(z^{\pm})^2/2$ in the wave: The main (+) component follows first the R^{-1} WKB law, but departs from it as soon as the z^- component becomes significant, which occurs earlier for the oblique case, as expected. Note that E^- reaches levels largely above the linear prediction mentioned above, which is here $E^-/E^+ \approx (\varepsilon/8)^2 \approx 1.4 \times 10^{-3}$. The waves evolve into both compressive and rotational discontinuities. Figure 3 shows, for the parallel case with $M=0.75$, the profile of the normalized velocity-magnetic field correlation $\sigma_c=(E^+-E^-)/(E^++E^-)$ ($\sigma_c=1$ initially). The sharp step in σ_c which

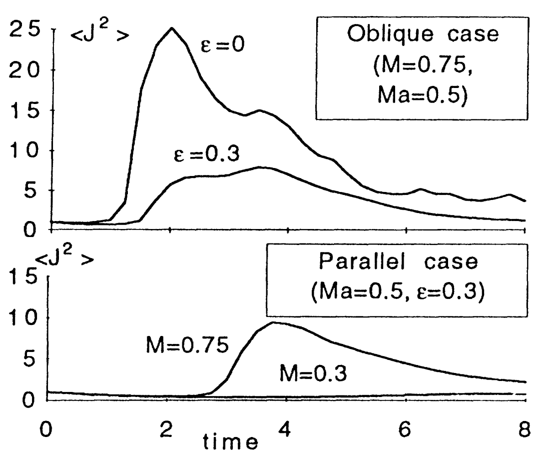

FIG. 4. Evolution of mean square current J^2 with time for circularly polarized Alfvén waves.

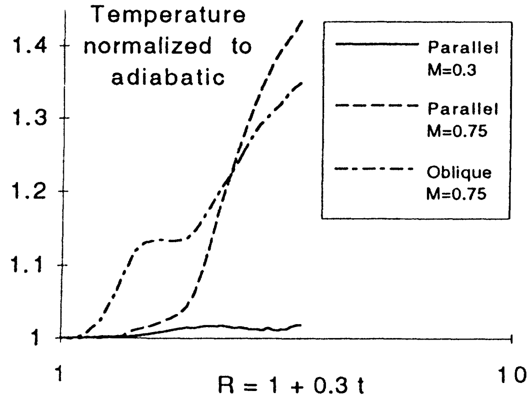

FIG. 5. Evolution with heliocentric distance of the "normalized temperature" $R^{10/3}\langle P\rangle/P^0$, showing deviation from the adiabatic decay.

forms (visible at $t=5$ and 8) corresponds essentially to a left-propagating fast compressive mode, while the dip (just right of the step at $t=5$) coincides with a rotational discontinuity. Note that the global decrease in σ_c remains limited, as $\sigma_c(t=8)=0.86$. Although the expansion causes nonlinear steepening of the initially parallel wave (significantly only when $M=0.75$), its main effect is to *delay* the evolution of oblique waves. This is apparent by comparing, for the expanding and nonexpanding oblique cases, the growth of the mean square current (top of Fig. 4), which is an indicator of energy transfer to small scales: The current maximum is reached in a longer time when $\varepsilon=0.3$ than when $\varepsilon=0$, and its magnitude is also smaller. We also show for completeness the growth of mean square current in the parallel case (bottom of Fig. 4). The growth of current (as well as of the velocity gradients) in turn leads to a "turbulent" heating which makes the temperature decay slower than the $R^{-4/3}$ adiabatic law (see Fig. 5, where the temperature evolution is normalized to adiabatic). The heating is negligible for the parallel case with $M=0.3$, but quite a significant departure from adiabatic is seen for $M=0.75$, both for the parallel and oblique cases.

Our results on oblique waves and also on the simple case of Burgers equation thus indicate that in general the nonlinear cascade of turbulence to small scales is delayed and possibly asymptotically frozen. However, even when starting from a parallel circularly polarized Alfvén wave, we obtain at $M=0.75$ and $M_A=0.5$ enough small scale excitation and thus enough heating to significantly reduce the temperature decrease with distance as compared to the adiabatic. The fact that the nonlinear decay of Alfvén waves of solar origin release a significant amount of heat only at large Mach number is reminiscent of the solar wind situation: higher Mach numbers are indeed observed within fast streams [3] which also show a less than adiabatic decay [1]. On the other hand, we also obtain a larger z^- component when the Mach number is high, which is contrary to the observations [3]. This may be related to initial conditions near the sun [2], but may also have a dynamical origin, related to the large scale velocity and magnetic shear structures which are not taken into account here (see [10]). It will be interesting to incorporate these large-scale velocity shear within the expansion: this will necessitate two- or three-dimensional simulations of Eq. (1), at high enough resolution to observe turbulent dissipation.

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