Berry's Phase and the Magnus Force for a Vortex Line in a Superconductor

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We show that the existence of the Magnus force is a general property of a vortex line in a superconductor by calculating the Berry phase for an adiabatic motion around a closed loop at zero temperature. We find that there is no influence of the disorder and the electromagnetic field on the existence of the Magnus force in the superconducting state, and its magnitude is proportional to the superfluid electron density.

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The motion of a vortex line plays an essential role in the understanding of many properties of superconductors [1]. The problem of the existence of the Magnus force acting on a single moving vortex line in a superconductor is, however, an unsolved one [2]. The advance in the past decade in material science makes it possible to perform a quantitative study of dynamical effects such as the quantum decay of supercurrent [3], the Hall effect [4], and the anomalous Hall effect [5] in a superconductor. Because of its important role played in those effects, the problem of the existence of the Magnus force has gained a renewed interest recently [3–6]. The argument for the existence of the Magnus force was first advanced by Friedel, de Gennes, and Matricon [7]. However, the opposite conclusion that there is no Magnus force in a superconductor was reached by different authors [8,9]. It was pointed out by Bardeen [8] that there is an error in the argument of Friedel, de Gennes, and Matricon [7]. Interestingly, both points of view seem to have substantial experimental support. By noting that in experiments one must consider the details of pinning and friction, this controversial issue was resolved by Nozières and Vinen [10]. They found that a proper account of those details renders practically no difference between the two opposite theories in the explaining of available experiments at that time. Nevertheless, Nozières and Vinen [10] endorsed the existence of the Magnus force by applying the classical ideal fluid results to a superconductor. In addition to this argument for the existence of the Magnus force, it has been shown that if the time-dependent Ginzburg-Landau (TDGL) equation takes the form of the nonlinear Schrödinger equation there will be a Magnus force [6]. Although the Magnus force is believed to be a general property of a vortex line [11], none of the above known phenomenological arguments in favor of the existence of the Magnus force for a vortex line in a superconductor is satisfactory.

Because there is no well-controlled microscopic derivation of the TDGL equation near zero temperature, any conclusion about the Magnus force based on a TDGL equation will be questionable. In fact, conflicting results based on different forms of TDGL equations exist [3,6,9]. The argument of Nozières and Vinen [10] is not only phenomenological, but is also valid only in the clean and extreme type-II superconductor limit, where there is no disorder and the influence of the electromagnetic field is negligible. As in a superconductor a quantized magnetic flux is always associated with a vortex line and the disorder normally exists, the argument of Nozières and Vinen [10] is particularly unsatisfactory. Because of the lack of a microscopic derivation of the Magnus force and a clear treatment of the influence of the disorder and the electromagnetic field, the doubt on the existence of the Magnus force still exists [2]. Given the important role played by the Magnus force in numerous effects in a superconductor [3–5], as well as important consequences associated with the Magnus force such as the Kelvin mode, the circular vibration along a vortex line [7], a clear answer to the important question of the existence of the Magnus force is needed. The purpose of the present paper is to provide such an answer at zero temperature. In this paper we find that the existence of the Magnus force is a general property of a vortex line, and is not influenced by the presence of the disorder and the electromagnetic field. In the following, we give a microscopic derivation of the Magnus force by calculating the associated Berry phase for an adiabatic motion.

For clarity we consider a two-dimensional superconducting film at zero temperature. The film is taken to be in the x-y plane, at rest in the laboratory frame. The argument can readily be generalized to the three-dimensional case. If the film is nonuniform, the energy of the vortex may depend on its position. Initially we take the large κ limit in which the modification of the magnetic field by the vortex is negligible.

We exploit the analogy between the motion of a quantized vortex in a superfluid film and the motion of the guiding center (the instantaneous position of the center of its cyclotron orbit) of a charged particle in a plane perpendicular to a strong magnetic field on which there is some varying potential energy $V(X,Y)$, but the argument can be developed without the use of this analogy. The mass of the charged particle is decoupled from the guiding center motion. The guiding center moves along equipotentials at a speed proportional to the potential gradient, so that the average Lorentz force balances the external force. The classical Lagrangian contains both $-V(X,Y)$ and a term linear in the velocity, and the wave function has a phase which consists of two parts: the dynamical phase which is the time integral of $-V(X,Y)/\hbar$, and the Berry phase [12], which is the path-dependent integral of the linear term. The Berry phase $\Delta\Theta$ round a closed loop is proportional to the magnitude of the magnetic field B and the area S enclosed by the loop projected in the direction of the magnetic field,

$$\Delta\Theta = 2\pi(q/h)BS\,, \tag{1}$$

with q the charge of the particle and h the Planck constant. This is 2π times the number of flux quanta enclosed by the loop.

The motion of a vortex can be described in similar terms. A vortex moves under the combined influence of the superfluid velocity fields produced either externally or by other vortices and of its position-dependent potential energy, so that the Magnus force, given by the vector product of the vorticity and the motion relative to the superfluid, balances the external force due to inhomogeneity of the film (pinning centers). The classical Lagrangian contains both the inhomogeneity potential and a term linear in the velocity of the vortex, and the wave function has both a dynamical phase and the Berry phase which is the integral of the linear term. The Berry phase round a closed loop is 2π times the number of superconducting electron pairs enclosed by the loop. In both of these examples a total derivative can always be added to the linear term in the Lagrangian (and to the Berry phase) without changing the equations of motion; the integral round a closed loop remains invariant.

Both the dynamical phase and the Berry phase [12] associated with a vortex in a superconductor can be calculated from the many-body wave function which describes the state of a vortex. Let $\Phi_0(\mathbf{r}_1,\ldots,\mathbf{r}_N)$ be the many-body wave function of the superconductor in the absence of a vortex, either for the ground state or for some nonequilibrium state with a nonzero superfluid velocity distribution $\mathbf{v}_s(\mathbf{r})$. Here N is the total number of electrons and $\{\mathbf{r}_j\}$ are the positions of electrons. The many-body wave function is antisymmetrized and is normalized to unity, $\int\prod_{j=1}^{N} d^2\mathbf{r}_j|\Phi_0|^2=1$. According to the work of London [13], with the correction made by Brenig [14] to account for the Cooper pairing in the superconductor, the desired many-body trial wave function Ψ_v for a vortex at position $\mathbf{r}_0$ is

$$\Psi_v(\mathbf{r}_1,\ldots,\mathbf{r}_N;\mathbf{r}_0) = \exp\left[\frac{i}{2}\sum_{j=1}^{N}\theta(\mathbf{r}_j-\mathbf{r}_0)\right]\Psi_0(\mathbf{r}_1,\ldots,\mathbf{r}_N;\mathbf{r}_0)\,, \tag{2}$$

where $\theta(\mathbf{r})=\arctan(y/x)$ and Ψ_0 is close to Φ_0, with modifications to describe the reduction in superfluid density near the vortex core, correlations induced by the flow pattern further from the core [15], and various other effects. It is normalized to unity as well as antisymmetrized. The fact that the phase θ is divided by 2 is the manifestation of the Cooper pairing in a superconductor. We note that a similar form of the many-body wave function to describe the vortex state in the case of superfluid helium has been used [16,17], where Ψ_0 is instead symmetrized.

The dynamical phase associated with the vortex corresponds to the extra energy it carries. For the case in which Φ_0 is the ground state this can be written as

$$\frac{\hbar}{2m}\int dt\int d^2\mathbf{r}\,\rho(\mathbf{r},\mathbf{r}_0)[\nabla\theta(\mathbf{r}-\mathbf{r}_0)]^2\,, \tag{3}$$

where m is the electron mass. The superfluid electron number density

$$\rho(\mathbf{r},\mathbf{r}_0)=\rho_s(\mathbf{r})+\delta\rho(\mathbf{r},\mathbf{r}_0) \tag{4}$$

is the sum of its value ρ_s in the absence of the vortex and the modification due to the presence of the vortex. For a nonuniform system the value of this integral depends on the position of the vortex, and so gives rise to a potential energy of the vortex, whose gradient gives the pinning force. In the presence of an initial superfluid flow field in Φ_0 there is an additional term

$$\int dt\int d^2\mathbf{r}\,\rho(\mathbf{r},\mathbf{r}_0)[\nabla\theta(\mathbf{r}-\mathbf{r}_0)]\cdot\mathbf{v}_s(\mathbf{r}) \tag{5}$$

in the dynamical phase.

If the vortex position $\mathbf{r}_0$ moves adiabatically along a path Γ, the Berry phase $\Delta\Theta_v$ is [12]

$$\Delta\Theta_v(\Gamma) = -\operatorname{Im}\int_\Gamma d\mathbf{r}_0\cdot\langle\Psi_v|\nabla_{\mathbf{r}_0}\Psi_v\rangle\,. \tag{6}$$

Performing the same calculation as in the cases of superfluid helium [18] and the fractional quantum Hall effect [19], we find the Berry phase as

$$\Delta\Theta_v(\Gamma) = -\int d^2\mathbf{r}\int_\Gamma d\mathbf{r}_0\cdot[\nabla_{\mathbf{r}_0}\theta(\mathbf{r}-\mathbf{r}_0)]\frac{\rho(\mathbf{r},\mathbf{r}_0)}{2}\,. \tag{7}$$

We substitute Eq. (4) for the superfluid density into this. The contribution of $\delta\rho$ to the Berry phase is a finite constant for the closed loop larger than the size of the vortex core and for our present purpose it can be ignored. If the background superfluid density ρ_s is constant, from Eq. (7) we find the Berry phase for a closed loop as follows:

$$\Delta\Theta_v(\Gamma) = -2\pi(\rho_s/2)S(\Gamma)\,, \tag{8}$$

with S the area enclosed by the loop Γ. We note that the Berry phase is 2π times the total number of superconducting electron pairs enclosed by the loop. This gives the result argued on classical grounds, and is directly analogous to Eq. (1) for electrons in a strong magnetic field, with the number of flux quanta replaced by the number of superconducting electron pairs.

The Magnus force is obtained from the derivatives of the two parts of the phase linear in the displacement of the vortex core, given by Eqs. (5) and (8). Taken together these give

$$\mathbf{F}_m = q_v(\mathbf{v}_s - \dot{\mathbf{r}}_0) \times \hat{\mathbf{z}} h \frac{\rho_s}{2} , \tag{9}$$

with $q_v = 1$ (-1) for a vortex parallel (antiparallel) to the z direction and $\dot{\mathbf{r}}_0$ the vortex velocity. This is identical to the one in classical fluid dynamics [11]. We emphasize that the Magnus force explicitly depends on the number density instead of the mass density of electrons. The effect of the lattice has been included through its effect on the value of ρ_s.

In the absence of disorder ρ_s is constant, this is the only force on the system, and it must be zero, so that the vortex moves with the local superfluid velocity. In the presence of disorder, at zero temperature, the Magnus force must balance the pinning force. At higher temperatures the presence of frictional forces acting on the vortex must also be taken into account [10].

In the above derivation we have actually only used the following two basic facts: the single valuedness of wave function and the finite density of the superfluid electrons. Those two facts are well represented by the many-body wave function of Eq. (2). The details of the many-body wave function are irrelevant here. This observation suggests that the result of Eq. (8) should have a much wider valid regime than the above neutral case in the clean limit. We here demonstrate that Eq. (8), and therefore Eq. (9), are indeed valid in the case of finite homogeneous disorder, and later we shall show that it is also valid in the real superconductor with the effect of the electromagnetic field. If the disorder is not extremely strong, the superconducting state exists at zero temperature [20]. A many-body wave function Φ_0 can again be used to describe the superconducting state without a vortex, although it is profoundly influenced by the disorder. Then the many-body trial wave function Ψ_v of the form of Eq. (2) can still be used to describe a vortex state. In the presence of finite disorder some electronic states, a fraction of Cooper pairs, will become localized [20] and will not be able to contribute to the supercurrent. The superfluid electron density ρ_s is then decreased. By allowing the superfluid electron density to be disorder dependent, we reach the conclusions that the Magnus force in the form of Eq. (9) remains unchanged in a dirty superconductor and its magnitude is reduced because of the reduction of the superfluid electron density by disorder.

Now we consider the real superconductor by putting the coupling to the electromagnetic field back into the problem. The many-body wave function Ψ_v of Eq. (2) is still the correct description of a vortex state in the presence of the electromagnetic field [13,14]: It is obviously single valued. Starting from it we can calculate the electric current, then the magnetic field according to Maxwell's equations, and find the magnetic flux associated with the vortex to be the magnetic flux quantum $h/2e$. Therefore, performing exactly the same calculation leading to Eq. (9) we find that the Magnus force remains unchanged. In an influential review article [21] the first term on the right-hand side of Eq. (9) is called the Lorentz force, and the second term the Magnus force. While the so-called Lorentz force is generally accepted, the second term is not [2]. The present demonstration shows that both terms exist and the physics behind them is the same: the dynamical and Berry phases for an adiabatic motion of a vortex. It also suggests that the name Lorentz force used in Ref. [21] is improper because the Magnus force is not a consequence of electromagnetic effects on a vortex.

Although the electromagnetic field has no effect on the Magnus force, we make one comment concerning its subtle effect on the many-body wave function of a vortex. As the vortex moves adiabatically along a closed loop, because of the magnetic flux associated with it, an additional phase will be picked up according to the Aharonov-Casher effect [22] if there is any electric charge inside the loop. However, because of the charge neutrality of the system, the Aharonov-Casher phases from electrons and the background will completely cancel each other, and no influence on the Magnus force will be found. Nozières and Vinen [10] also reached the conclusion of no influence from the electromagnetic field on the Magnus force. However, as we have pointed out in the second paragraph, they assumed the condition of the extreme type-II superconductor limit. Consequently, they found that the influences both from electrons and background are negligible, not the cancellation found here. On the other hand, the complete cancellation between the contributions to the Magnus force from electrons and the positive background for an adiabatic motion of a vortex has been pointed out by Bardeen [8] by using Faraday's law. However, he reached the incorrect conclusion that there is no Magnus force by treating the Magnus force incorrectly as an electromagnetic force.

The foregoing general demonstration would suggest that the Magnus force also exists for a vortex line in a normal Fermi liquid. While this is formally correct, it is not well enough defined to discuss it in a normal Fermi liquid because there a vortex state is highly unstable. In a superconductor the vortex state is instead very stable because of the presence of the condensate [23,24]. Therefore vortices in a superconductor behave as stable particles, which is essential for the present discussion of the Magnus force.

In summary, starting from the many-body wave function description of a vortex in a superconductor at zero temperature, we have derived the Magnus force by the calculation of the Berry phase when a vortex moves adiabatically along a closed loop, in analogy to the case of a charged particle in the presence of a magnetic field. We are able to show that the existence of the Magnus force relies only on two basic properties of a superconductor: the existence of the superconducting state and the single valuedness of the many-body wave function. Therefore we have found that the existence of the Magnus force is insensitive to the details of the system and its magnitude is proportional to the superfluid electron density.

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