General Linear Mode Conversion Coefficient in One Dimension

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A general formula is presented for the mode conversion coefficient for linear mode conversion in one dimension, in terms of an arbitrary 2×2 reduced dispersion matrix describing the coupling of the modes. The mode conversion coefficient has three invariance properties which are discussed, namely, invariance under scaling transformations, canonical transformations, and a certain kind of Lorentz transformation. Formulas for the S matrix of mode conversion are also presented. The example of the conversion of electromagnetic waves to electrostatic waves in the ionosphere is used to illustrate the formulas.

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Linear mode conversion is a fundamental process by which linear waves interact. It is of great importance in plasma physics, and is central to a number of physical processes. Important work in this area in recent years has included studies by Cairns and Lashmore-Davies [1], Fuchs and Bers [2], Friedland [3], Friedland and Kaufman [4], Hickel-Lipsker, Fried, and Morales [5], Mjølhus and Flå [6], Williams [7], Friedland, Goldner, and Kaufman [8], Kaufman and Friedland [9], Tracy and Kaufman [10], Romero and Scharer [11], and Kull, Kashuba, and Berk [12], and a review has been written by Stix and Swanson [13]. The field has been very active, and what we offer here is only a representative list.

In this paper we report general formulas for the mode transmission and conversion coefficients T and C and the S matrix which represent the coupling of two linear modes of oscillation of a plasma. Our formulas give C, T, and S as explicit functions of the (reduced) dispersion matrix in the mode conversion region. Previously, these quantities have been worked out only for dispersion matrices of special forms; typical assumptions have been that the off-diagonal elements are small and constant, or that one of the components of the dispersion matrix is small and slowly varying in both x and k. Our formulas, on the other hand, remove all such assumptions.

The assumptions we do make are the following. The first is that the plasma has a spatial variation in only one direction, which is what we mean by "one dimension" in the title of this Letter. Slab models are allowed under this designation. The second assumption is that mode conversion actually occurs. This is equivalent to assuming that two of the eigenvalues of the dispersion matrix are small in some region of the x-k phase plane. Third, we assume that the dispersion matrix is Hermitian. This is an assumption of mathematical convenience which we plan to remove in the future; for the time being it restricts the applicability of our methods to cold fluid and certain other models.

To emphasize the generality of our methods, we will outline how one can extract a mode conversion problem out of an essentially arbitrary set of coupled linear wave equations. We begin with a wave equation of the form $\hat{D}_{\alpha\beta}\psi_\beta=0$. As an example, ψ_α for $\alpha=1,2,3$ might be the electric field and $\hat{D}_{\alpha\beta}$ the usual 3×3 dispersion matrix $D_{\alpha\beta}(x,k)$, promoted into a linear operator by replacing k by $-i\partial/\partial x$; but actually our methods can be applied to general systems of coupled wave equations, so we allow ψ_α to stand for any vector of field variables, $\hat{D}_{\alpha\beta}$ to stand for any matrix of linear wave operators, and $D_{\alpha\beta}(x,k)$ to stand for the corresponding generalized dispersion matrix, a matrix of functions of x and k. [The caret stands for a linear operator, and distinguishes the matrix of wave operators $\hat{D}_{\alpha\beta}$ from the dispersion matrix $D_{\alpha\beta}(x,k)$.] We assume that the plasma is stationary and that the dispersion matrix depends implicitly on the frequency ω.

When the wave equation $\hat{D}_{\alpha\beta}\psi_\beta=0$ is solved by WKB theory [14,15], the waves can be visualized as living on the "dispersion curves" $\lambda(x,k)=0$ in the x-k phase plane, where $\lambda(x,k)$ is an eigenvalue of $D_{\alpha\beta}(x,k)$. The waves have the form $\psi_\alpha(x)=A_\alpha(x)E^{iS(x)}$ where A_α is the vector amplitude and S is the rapidly varying phase. But the WKB solution breaks down if two dispersion curves come close together in some region of the phase plane, as illustrated schematically by the dotted circle in Fig. 1. We refer to such regions as "mode conversion regions"; in such regions, the dispersion curves are approximated by hyperbolas, and WKB theory is not valid because the

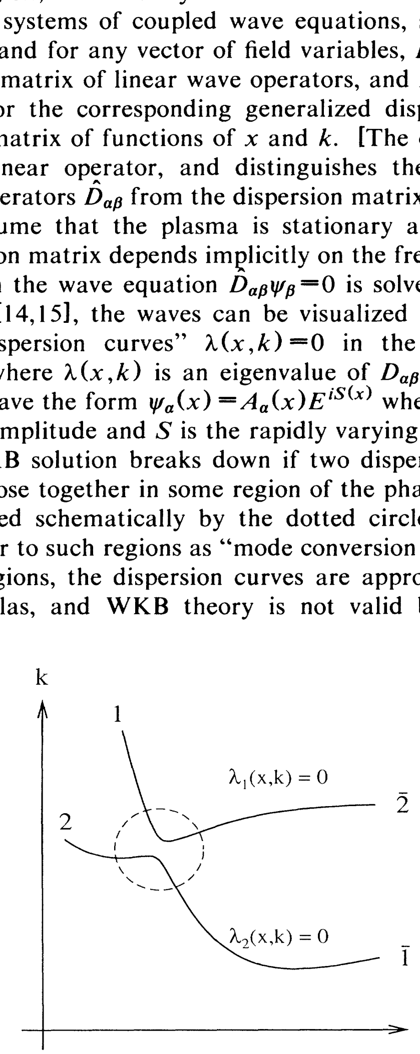

FIG. 1. Schematic illustration of a typical mode conversion in the x-k phase plane in one dimension.

eigenvectors of the dispersion matrix are rapidly varying. This point has been discussed by Friedland and Kaufman [4] and ourselves [15]. In the mode conversion regions, two eigenvalues of $D_{\alpha\beta}(x,k)$ are small, since the curves where two λ's vanish come close together. WKB theory is valid outside the mode conversion region; there are four branches in this outer region, labeled $1,2,\bar{1},\bar{2}$ in Fig. 1, on which WKB waves live. These branches are coupled within the mode conversion region itself.

We now define the mode transmission and conversion coefficients and the S matrix. We write $c_i\psi_{\alpha i}(x)$, $i=1$, $2,\bar{1},\bar{2}$, for the four WKB waves entering or exiting the mode conversion region, where the c_i's are constant coefficients. We imagine that waves 1 and 2 enter the mode conversion region, and waves $\bar{1}$ and $\bar{2}$ exit. The amplitudes A_α of these four branches are normalized to represent unit action flux, i.e., $\sum_\alpha |A_\alpha|^2 \partial\lambda/\partial k=1$, and certain phase conventions are observed which we will describe in forthcoming publications. Then in terms of the c_i's, we define the S matrix by

$$\begin{pmatrix} c_{\bar{1}} \\ c_{\bar{2}} \end{pmatrix} = \begin{pmatrix} S_{\bar{1}1} & S_{\bar{1}2} \\ S_{\bar{2}1} & S_{\bar{2}2} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \tag{1}$$

so that the components of S specify both the amplitudes and phases of the coupling. Finally, we define the mode transmission and conversion coefficients by $T=|S_{\bar{1}1}|^2$, $C=1-T$; C is the amount of action flux exiting on branch $\bar{2}$ if unit action enters on branch 1 and zero action enters on branch 2. The S matrix is unitary, and satisfies $T=|S_{\bar{2}2}|^2=|S_{\bar{1}1}|^2$.

Now we assume that two eigenvalues of $D_{\alpha\beta}(x,k)$, say, $\lambda_1(x,k)$ and $\lambda_2(x,k)$, are both small in some region of the phase plane. This is the "mode conversion region," and we define a certain point (x_0,k_0) in this region, the "mode conversion point," as the point where the product $\lambda_1\lambda_2$ is stationary with respect to both x and k. That is, (x_0,k_0) is the root of

$$\partial(\lambda_1\lambda_2)/\partial x=\partial(\lambda_1\lambda_2)/\partial k=0\,. \tag{2}$$

It turns out that (x_0,k_0) is at the crossing of the asymptotes of the hyperbolas formed by the dispersion curves; this definition generalizes the earlier definition of Cairns and Lashmore-Davies [1] for the mode conversion point.

In the mode conversion region, the coupling of the two modes is described by a 2×2 matrix, a reduced version of $\hat{D}_{\alpha\beta}$ or $D_{\alpha\beta}(x,k)$. An algorithmic process whereby an arbitrary system of coupled wave equations can be reduced to a 2×2 form has been described by Friedland and Kaufman [4]; we present here a simplified prescription, valid when one wants only the form of the reduced 2×2 matrix in the mode conversion region. Our prescription will be justified fully in future publications. Given an $N\times N$ matrix $D_{\alpha\beta}(x,k)$ and a mode conversion point (x_0, k_0) defined as above, we compute the orthonormal eigenvectors of $D_{\alpha\beta}(x_0,k_0)$ (at the mode conversion point) and transform $D_{\alpha\beta}(x,k)$ (at any point) to the basis given by these eigenvectors. Then the reduced 2×2 matrix in the mode conversion region is the submatrix of the original matrix corresponding to the two small eigenvalues. This matrix still depends on x and k; but since the transmission and conversion coefficients T and C and S matrix depend only on $D_{\alpha\beta}$ and its derivatives in the mode conversion region, it is often convenient to expand $D_{\alpha\beta}(x,k)$ to first order in x and k about the mode conversion point.

We have stated that our formulas for the mode transmission and conversion coefficients are more general than previous formulas which have been given; but such generality is not needed unless actual problems give rise to dispersion matrices which are not already in some standard form. We now give an example which shows that such generality is indeed required.

We consider the problem of electromagnetic waves transmitted from the surface of the Earth into the ionosphere, where they convert into electrostatic waves in the magnetized plasma at the resonance layer $\omega=\omega_p(x)$. This problem has an extensive literature [16–18], of which the recent work by Mjølhus is the most complete. We begin by summarizing our assumptions and notation for this problem. We place the x axis vertical, we place the y and z axes so that the Earth's magnetic field (assumed constant) has the direction $\hat{\mathbf{b}}=\hat{\mathbf{x}}\cos\alpha+\hat{\mathbf{z}}\sin\alpha$. We use a slab model with variation only in the x direction, we use a cold fluid model, ignoring the dynamics of the ions, we introduce the dimensionless parameters $X=\omega_p^2(x)/\omega^2$, $Y=|\Omega_e|/\omega$, and we write $\mathbf{N}=\mathbf{k}c/\omega$. We take the case $\omega>|\Omega_e|$, i.e., $Y<1$. The phase plane coordinates are (x,k_x), or, equivalently, (X,N_x), which are functions of one another along a ray. The quantities N_y and N_z depend on the launch angle and are constant along a ray. Mode conversion will not occur, i.e., two eigenvalues of the dispersion matrix will not be small in the same region of the phase plane, unless the launch angle is chosen so that when the ray passes near the resonance layer $X=1$, the vector $\mathbf{N}$ is close to $\mathbf{N}_c=A\hat{\mathbf{b}}$, where $A=[Y/(1+Y)]^{1/2}$. We assume the launch angle is so chosen, and we write $X=1+\xi$, $\mathbf{N}=\mathbf{N}_c+\mathbf{n}$ where ξ and n_x are small in the mode conversion region and n_y,n_z are small everywhere. We now regard (ξ,n_x) as the coordinates in the phase plane.

With these assumptions, mode conversion couples the left circularly polarized electromagnetic wave with the Langmuir wave, and the eigenvectors of the dispersion matrix at the mode conversion point are simply ($\hat{\mathbf{b}},\hat{\mathbf{e}}_+$, $\hat{\mathbf{e}}_-$), where $\hat{\mathbf{e}}_\pm=(\hat{\mathbf{y}}\pm\hat{\mathbf{b}}\times\hat{\mathbf{y}})/\sqrt{2}$. In this basis, the dispersion matrix nearly block diagonalizes in the mode conversion region, and the 2×2 subblock describing the mode conversion has the components

$$\begin{aligned} D_{11}&=-\xi/(1+Y)-2A(n_x\cos\alpha+n_z\sin\alpha)\,,\\ D_{12}&=A[n_y+i(n_x\sin\alpha-n_z\cos\alpha)]/\sqrt{2}\,,\\ D_{22}&=-\xi\,, \end{aligned} \tag{3}$$

with $D_{21}=D_{12}^*$. Notice that the off-diagonal elements are not small constants, nor are they slowly varying, since as we move away from the mode conversion region, n_x assumes values which are not small. We have neglected terms quadratic in ξ, $\mathbf{n}$ in these expressions, so the formulas given are valid only in the mode conversion region. We now return to our presentation of the transmission coefficient for a reduced dispersion matrix of arbitrary form.

Before writing down the formula for the transmission coefficients T, we note some invariance properties which it must have. First, we note that T cannot change if $D_{\alpha\beta}$ is subjected to a scaling transformation, $D_{\alpha\beta}\to cD_{\alpha\beta}$, where $c\neq 0$ is a constant, because such a transformation is equivalent to simply multiplying the wave equation $\hat{D}_{\alpha\beta}\psi_\beta=0$ through by c, with no effect on the physics. Therefore T must be a homogeneous function of degree 0 of $D_{\alpha\beta}$ and its derivatives. Next, the transmission coefficient T cannot depend on the canonical coordinates used in the phase plane, a fact which has previously been explored by Friedland, Goldner, Kaufman, and Tracy [8-10]. This is because action fluxes in WKB theory are independent of canonical coordinates. The invariance or covariance of WKB theory under canonical transformations was explored in an early paper by Miller [19], and since that time has been the subject of deep investigations by mathematicians [20]. In the present case, it means that the formula for T in terms of $D_{\alpha\beta}$ must have the same form if (x,k) are replaced by any new variables (x',k') which are related to (x,k) by a canonical transformation. This in turn implies that any derivatives with respect to x and k which appear must be expressible in terms of Poisson brackets.

A final invariance property of the formula for T follows from the fact that the transmission coefficients cannot change if we mix the two wave fields in the wave equation $\hat{D}_{\alpha\beta}\psi_\beta=0$, which describes the two coupled modes. That is, if we write $\psi_\alpha=Q_{\alpha\beta}\psi'_\beta$, where ψ and ψ' are respectively the old and new wave fields and $Q_{\alpha\beta}$ is some constant, possibly complex, invertible, 2×2 matrix, then we will have transformed the wave equation into $\hat{D}'_{\alpha\beta}\psi'_\beta=0$, where the new dispersion matrix is given by $D'=Q^\dagger DQ$ (we have multiplied the wave equation on the left by $Q^\dagger$ so that D' will be Hermitian). Then the form of the formula for the transmission coefficient must be the same, whether we use D or D'.

It turns out this final invariance property is equivalent to invariance under a kind of Lorentz transformation. These Lorentz transformations have nothing to do with relativity theory in a physical sense, but have the same mathematical structure as physical Lorentz transformations. To show how this comes about, we parametrize the four components of $D_{\alpha\beta}$ by a real four-vector B^μ, $\mu=0,1,2,3$, where $B^0=(D_{11}+D_{22})/2$, $B^1=(D_{12}+D_{21})/2$, $B^2=i(D_{12}-D_{21})/2$, $B^3=(D_{11}-D_{22})/2$, or, compactly, $D=B^\mu\sigma_\mu$, where $\sigma_0=I$ and σ_i, $i=1,2,3$, are the Pauli matrices. Next, when we perform the transformation $D'=Q^\dagger DQ$, we demand that $\det Q=+1$, because if $\det Q\neq 1$, we can multiply by a constant to make $\det Q=1$. (The constant gives us a trivial scaling transformation, discussed above.) In group theoretical language, we have restricted Q to the group SL(2,$\mathbb{C}$). Then when we transform D according to $D'=Q^\dagger DQ$, it turns out that the four-vector B^μ transforms according to a Lorentz transformation, $B'^\mu=\Lambda^\mu_\nu B^\nu$, where Λ is a 4×4 real Lorentz transformation matrix which depends on Q. This fact is well known in relativistic quantum mechanics [21], in which Q matrices of the kind we have described are used to perform Lorentz transformations on spin $\frac{1}{2}$ particles. The significance of these facts for our purposes is that if the transmission coefficient T is expressed in terms of the four-vector B^μ, then it must be manifestly invariant under Lorentz transformations.

These three invariance properties, scaling, canonical, and Lorentz invariance, almost uniquely determine the formula for T. The actual formula, which is one of our principal results, is $T=\exp(-2\pi\gamma)$, where

$$\gamma=\frac{|B_0^\mu B_{0\mu}|}{|2\{B^\mu,B^\nu\}\{B_\mu,B_\nu\}|^{1/2}}\,. \tag{4}$$

In this formula, B_0^μ is the four-vector corresponding to $D_{\alpha\beta}(x_0,k_0)$, indices are raised and lowered as in relativity theory with the metric $g_{\mu\nu}=\mathrm{diag}(+1,-1,-1,-1)$, and the curly brackets are the x-k Poisson brackets, which are evaluated at the mode conversion point (x_0,k_0). Furthermore, the S matrix is given by $S_{\bar{1}1}=-S_{\bar{2}2}=-e^{-\pi\gamma}$, $S_{\bar{1}2}=S_{\bar{2}1}^*=e^{-i\phi}C^{1/2}$, where $\phi=\pi/4+\arg\Gamma(i\gamma)+\gamma-\gamma\ln\gamma$. The quantity γ is the simplest scalar one can form from $B^\mu(x,k)$ with the required invariance properties; thus it is no surprise that T is a function of γ. The actual functional form is obtained in the usual way, by solving parabolic cylinder equations. In our formula for γ, the derivatives of $D_{\alpha\beta}$ with respect to x and k are equivalent to the group velocities and equilibrium gradients seen in the formula given by Cairns and Lashmore-Davies [1].

We return now to our example, given by the dispersion matrix in Eq. (3), and first ask for the mode conversion point as specified by Eq. (2). We note that $\lambda_1\lambda_2=\det D=B^\mu B_\mu$, and perform a short calculation to find the coordinates of the mode conversion point, $\xi_0=-A(1+Y)n_z\times\sin\alpha/d$, $n_{x0}=-(1+2Y)n_z\sin\alpha\cos\alpha/d$, where $d=1+(1+2Y)\cos^2\alpha$. It is then straightforward to calculate γ from Eq. (4); we find, in agreement with Mjølhus [18],

$$\gamma=(\omega/cX')(Y/8d)^{1/2}[n_y^2+2(1+Y)n_z^2/d]\,, \tag{5}$$

where $X'=dX/dx$ evaluated at $X=1$. Mjølhus derived this result by using special methods (i.e., methods which do not obviously generalize to other dispersion matrices), and he treated several different cases separately.

In conclusion, we mention three further points. First, singularities of dispersion matrices of the form $1/[\omega$

$-\Omega(x)]$ are subsumed under our formalism, and Eq. (4) applies. In the neighborhood of such singularities, the dispersion curves behave like $k=1/x$, which are hyperbolas as in Fig. 1. A 45° rotation in phase space, as used by Tracy and Kaufman [10], converts the corresponding wave functions into parabolic cylinder functions. Our formula (4), being a canonical invariant, does not care about the orientation of the hyperbolas. Second, many physical processes involve multiple mode conversions, sometimes occurring at a single spatial point. As long as these are separated in phase space, they can be treated individually, with ordinary WKB propagation in between (possibly in k space). Thus, various reflection coefficients can be computed by compounding elementary S matrices as given here. Third, our transformation theory is important for mode conversion in higher dimensions. Such problems have previously been studied in cases of high symmetry, but very little is known about generic cases.

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