Stabilization of the No-Motion State in Rayleigh-Bénard Convection through the Use of Feedback Control

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(Received 14 August 1992)

It is demonstrated theoretically that the critical Rayleigh number for transition from the no-motion (conduction) to the motion state in the Rayleigh-Bénard problem of an infinite fluid layer heated from below and cooled from above can be significantly increased through the use of feedback control strategies effecting small perturbations in the boundary data.

PACS numbers: 47.27.Cn, 47.20.Ky, 47.27.Te

Rayleigh-Bénard convection occurs in a horizontal fluid layer heated from below and cooled from above. As the temperature difference between the layer's bottom and top (expressed in non-dimensional form as the Rayleigh number) increases, the state of the system undergoes a sequence of bifurcations from no-motion to time-independent, cellular motion to time-dependent (possibly periodic) motion to chaos and turbulence.

The Rayleigh-Bénard problem has been studied extensively in the physics and engineering literature [1]. From the fundamental point of view, this problem is of interest as it provides an experimentally realizable system for studies of nonlinear phenomena in dissipative systems such as pattern formation, bifurcation sequences, and the transition to turbulence. From the practical point of view, this problem is a paradigm for many important natural and technological processes such as convection in the atmosphere, in the oceans, in stars, and in the melt of solidification and crystal growth processes.

In many situations, it may be advantageous to alter the normally occurring bifurcation sequence. For example, to improve crystal quality in crystal growth processes, it may be desirable to maintain the no-motion conductive state for Rayleigh numbers (R) far exceeding the critical one for the onset of convection ($R_c \sim 1707.76$).

In previous experimental and theoretical work [2], we were able to demonstrate that the bifurcation structure of simple convective systems exhibiting temporally complex behavior such as the thermal convection loop (which can be considered an experimental analog of the Lorenz equations [3]) can be controlled. In this Letter, we wish to extend similar ideas to the control of the bifurcation structure of the Rayleigh-Bénard problem, which exhibits both spatial and temporal complexity. More specifically, this Letter focuses on the use of feedback control to delay the transition from the no-motion state to the motion state.

Let us begin by considering an infinite, horizontal, Boussinesq's fluid layer. The layer is oriented so that its horizontal boundaries are normal to the gravity vector (which is parallel to the z coordinate). In the classical problem, the lower and upper boundaries ($z=\pm\frac{1}{2}$) are maintained at uniform temperatures with the bottom maintained at a temperature higher than the top. The temperature difference between bottom and top is the driving force. The conservation equations describing the fluid motion and the nondimensionalization scheme are given, for example, in Chandrasekhar [4] and for brevity's sake are not repeated here. The equations of motion admit a no-motion (conduction) solution. The stability of the no-motion state in the presence of small perturbations can be investigated using linear theory. The arbitrary perturbation quantities (deviations from the conductive solution) are expanded into a Fourier series whose components (normal modes) are of the form $f(z)\exp[i(k_x x - k_y y)+\sigma t]$, where x and y are horizontal Cartesian coordinates, t is time, k_x and k_y are wave numbers of the periodic disturbances in the x and y directions, respectively, and σ is the growth rate. For the no-motion state to be asymptotically stable, $\mathrm{Re}(\sigma)$ must be negative. The linearized equation for the temperature disturbance $T(z)$ of the classical problem is [4]

$$(D^2-a^2)(D^2-a^2-\sigma)(D^2-a^2-\sigma/\mathrm{Pr})T(z) = -a^2RT(z)\,. \quad (1)$$

In the above, $a^2=k_x^2+k_y^2$, $D=d/dz$, and Pr is the fluid's Prandtl number (the ratio between the kinematic viscosity and the heat diffusivity). The boundary conditions of the classical problem correspond, respectively, to zero temperature disturbance, zero normal velocity, and no slip at the horizontal boundaries ($z=\pm\frac{1}{2}$):

$$T=0\,, \quad (2)$$

$$(D^2-a^2-\sigma)T=0\,, \quad (3)$$

$$D(D^2-a^2-\sigma)T=0\,. \quad (4)$$

The classical problem of stability [Eqs. (1)-(4)] is self-adjoint and it can be shown that the growth rate σ is a real quantity (the principle of exchange of stability is valid). To solve the classical stability problem, one determines the smallest Rayleigh number for which the growth rate $\sigma=0$. Once the growth rate becomes positive, the no-motion solution is nonstable and will not be observed in experiments. Criticality occurs [4] when R_c

~ 1707.762 and $a_c \sim 3.117$. These values are independent of the Prandtl number.

Our objective is to maintain the no-motion state at Rayleigh numbers significantly exceeding the critical one (R_c) while we nominally maintain the same boundary conditions as in the original problem. To accomplish this objective, we propose using a number of sensors to measure the deviation of the fluid's temperature from its desired (conduction) value in a horizontal platform (i.e., $z=0$). The heat applied to the bottom boundary is slightly modified in proportion to this deviation. At locations where the flow is hotter than usual and the fluid tends to ascend, the boundary temperature is reduced to assist in dissipating the excess heat. The reverse occurs at locations where the fluid descends. Mathematically, this control strategy is manifested by modifying the boundary condition (2) to read

$$T(-\tfrac{1}{2}) = C(T(0)) \text{ and } T(\tfrac{1}{2}) = 0 . \tag{5}$$

In the above, $C(\cdot)$ is the controller's function. It is convenient to make C a linear function of its argument. For the purposes of this paper, we will use the proportional control:

$$T(-\tfrac{1}{2}) = -KT(0) \text{ and } T(\tfrac{1}{2}) = 0 , \tag{6}$$

where K is the controller's gain. The control strategy can be easily extended to include differential and integral controls. As our objective in this Letter is merely to demonstrate feasibility, we shall focus the discussion mostly on the proportional controller. We wish to determine the effect of the controller's gain K on the critical Rayleigh number at transition. To this end, we need to solve the modified eigenvalue problem consisting of Eqs. (1), (6), (3), and (4). In the modified problem, exchange of stability is not guaranteed and one cannot take $\sigma=0$ at criticality. For a chosen controller's gain (K), the solution procedure consists of the following steps: (i) Prescribe a value for the wave number a; (ii) search for values of R and $\mathrm{Im}(\sigma)$ so that $\mathrm{Re}(\sigma)=0$ and the equations are nontrivially satisfied; (iii) repeat the calculation for different a values. The critical Rayleigh number $R_{c,K} = \min_a R$.

Following Reid and Harris [5], we solve the system, (1), (6), (3), and (4) by substituting a solution of the form

$$T(z) = \sum_{i=1}^{3} \{(\mathrm{even})_i \cosh(x_i z) + (\mathrm{odd})_i \sinh(x_i z)\} , \tag{7}$$

where x_i ($i=1$, 2, and 3) are the positive square roots of the solutions of the cubic equation

$$(x_i^2 - a^2)(x_i^2 - a^2 - \sigma)(x_i^2 - a^2 - \sigma/\mathrm{Pr}) + a^2 R = 0 \tag{8}$$

for x_i^2. In (7), terms with coefficients (even) and (odd) correspond, respectively, to even and odd modes. Upon substituting (7) into the boundary conditions (6), (3), and (4), one obtains a set of six linear algebraic equations for the coefficients $(\mathrm{even})_i$ and $(\mathrm{odd})_i$. To assure that all these coefficients will not be identically zero, we need the system's determinant

$$\mathrm{Det}\begin{bmatrix} M & A \\ O & N \end{bmatrix} = 0 . \tag{9}$$

In the above, O is a 3×3 matrix with all its entries being zeroes,

$$M = \begin{bmatrix} \sinh\frac{1}{2}x_1 & \sinh\frac{1}{2}x_2 & \sinh\frac{1}{2}x_3 \\ (x_1^2 - a^2 - \sigma)\sinh\frac{1}{2}x_1 & (x_2^2 - a^2 - \sigma)\sinh\frac{1}{2}x_2 & (x_3^2 - a^2 - \sigma)\sinh\frac{1}{2}x_3 \\ x_1(x_1^2 - a^2 - \sigma)\cosh\frac{1}{2}x_1 & x_2(x_2^2 - a^2 - \sigma)\cosh\frac{1}{2}x_2 & x_3(x_3^2 - a^2 - \sigma)\cosh\frac{1}{2}x_3 \end{bmatrix}$$

corresponds to the odd modes, and

$$N = \begin{bmatrix} \frac{1}{2}K + \cosh\frac{1}{2}x_1 & \frac{1}{2}K + \cosh\frac{1}{2}x_2 & \frac{1}{2}K + \cosh\frac{1}{2}x_3 \\ (x_1^2 - a^2 - \sigma)\cosh\frac{1}{2}x_1 & (x_2^2 - a^2 - \sigma)\cosh\frac{1}{2}x_2 & (x_3^2 - a^2 - \sigma)\cosh\frac{1}{2}x_3 \\ x_1(x_1^2 - a^2 - \sigma)\sinh\frac{1}{2}x_1 & x_2(x_2^2 - a^2 - \sigma)\sinh\frac{1}{2}x_2 & x_3(x_3^2 - a^2 - \sigma)\sinh\frac{1}{2}x_3 \end{bmatrix}$$

corresponds to the even modes. As in the classical problem [4], the determinant (9) can be factored and one may seek the critical R and $\mathrm{Im}(\sigma)$ values which satisfy either $\mathrm{Det}[M]=0$ or $\mathrm{Det}[N]=0$. Clearly, the control strategy proposed here affects only the even modes. The critical Rayleigh numbers corresponding to the odd modes are reported in Chandrasekhar [4] and their smallest value is 17610.39 which is 10.3 times larger than the critical Rayleigh number of the uncontrolled ($K=0$), even modes.

Our objective is to determine the $R_{c,K}$ corresponding to $\mathrm{Det}[N]=0$ and $\mathrm{Re}(\sigma)=0$. To obtain these values, we employ the software package AUTO [6] which is capable of tracing bifurcation diagrams. The results of our computations are depicted in Figs. 1 and 2. Figure 1 depicts the normalized critical Rayleigh number at loss of stability $r = R_{c,K}/1707.762$ as a function of the wave number for the controller's gain $K=7$ and for Prandtl numbers $\mathrm{Pr}=0.1$, 0.5, 0.7, and 7. The solid and dashed lines correspond to loss of stability through a real eigenvalue $[\mathrm{Im}(\sigma)=\mathrm{Re}(\sigma)=0]$ and through an imaginary eigenval-

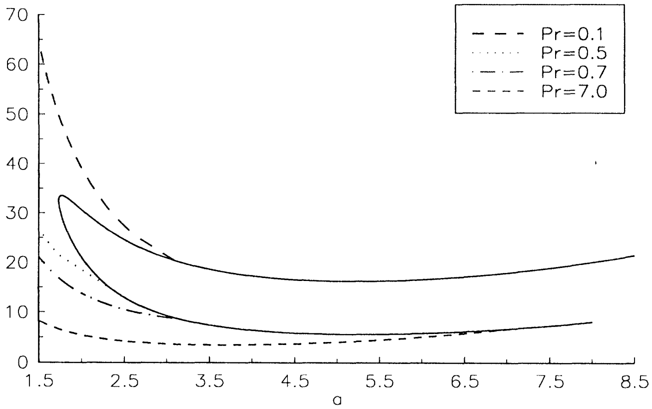

FIG. 1. The critical, normalized Rayleigh number (r) at the onset of convection is depicted as a function of the wave number a for controller's gain $K=7$ and for Prandtl numbers 0.1, 0.5, 0.7, and 7. The solid (independent of the Prandtl number) and dashed lines correspond to bifurcation through real and imaginary growth rates, respectively. The most dangerous mode (a_c) corresponds to $r_c = \min_a(r)$.

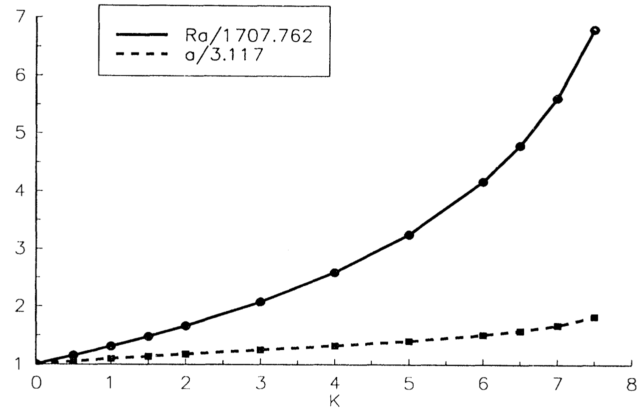

FIG. 2. The critical, normalized Rayleigh number (r) and the most dangerous mode (a) at the onset of convection are depicted as a function of the controller's gain K. Only bifurcations through a real growth rate are shown.

ue [$\mathrm{Im}(\sigma)\neq 0$, Hopf bifurcation], respectively. Intersections of the solid lines by the dashed ones correspond to triple-point bifurcations. As the Rayleigh number exceeds each of these points, the real part of three σ values changes simultaneously, from negative to positive. As is evident from Eq. (8), when the principle of exchange of stability is valid (along the solid line), the results are independent of the Prandtl number just as in the classical (uncontrolled) case. The Hopf bifurcation (dashed line), however, depends on the magnitude of the Prandtl number. As the Prandtl number increases, the Hopf bifurcation occurs at lower values of the Rayleigh number.

Figure 2 depicts the critical normalized Rayleigh number (r, the solid line) and the normalized wave number $a/3.117$ (dashed line) at criticality. The lines correspond to bifurcation through a simple eigenvalue. They are valid for fluids with $\mathrm{Pr} < 1$. For high Prandtl number fluids, only portions of the curve may be valid as the Hopf bifurcation into oscillatory convection (not shown in Fig. 2) may take place prior to the simple bifurcation (through a simple eigenvalue). For example, for a fluid with $\mathrm{Pr}=7$, the curve in Fig. 2 is applicable only for $K<6$. Figure 2 demonstrates that both the critical Rayleigh number at the onset of convection and the most dangerous wave number increase as monotonic functions of the controller's gain. There is no point in using much larger controller gains than those shown in Fig. 2 since soon thereafter one gets to Rayleigh numbers at which the odd modes are destabilized and the controller presented here has no effect on these modes.

An important issue not addressed in this Letter is the size of the basin of attraction of the stabilized no-motion state. In the classical problem, as long as $R<R_c$, the fluid motion will eventually decay regardless of disturbances' size. In other words, the no-motion state is globally attracting and the bifurcation into time-independent motion is supercritical. We do not yet have estimates of the size of the basin of attraction of the stabilized no-motion state. Based on numerical experiments, we can, however, make the following tentative observations. For small controller gains, i.e., $K<5$, the bifurcation into the motion state is supercritical and the no-motion state is globally attracting. For larger K values (i.e., $K>6$), one can identify three regimes: (i) For $R<R_G(K)$ [i.e., $R_G(7)\sim 8850$], the no-motion state is globally attracting. (ii) For $R_G(K)<R<R_c(K)$ [i.e., $R_c(7)=9580$], the domain of attraction is finite. Depending on the disturbances' magnitude and/or the initial conditions, the system will assume either a no-motion or a time-independent motion state. (iii) For $R>R_c$, the no-motion state becomes unstable with any disturbances. The establishment of the global stability characteristics of the controller requires further study.

The analysis assumed that the temperature distribution along the layer's midplane is known, that the bottom temperature can be continuously varied, and that the actuator's response is instantaneous. In practice, however, one may have to content oneself with a finite number of sensors and actuators and with a delayed sensor response. To test whether the no-motion state can still be stabilized with a finite number of sensors and actuators, we simulated the two-dimensional Oberbeck-Boussinesq equations numerically using central differences. The simulations were performed on a finite width medium with periodic boundary conditions. Although various widths were tested, in most of the simulations we used a width equal to the most dangerous wavelength, predicted by linear theory, for the chosen controller's gain. In the simulations, four temperature sensors were evenly distributed along the cell's midplane and the bottom surface was divided into four segments whose temperatures were uniform and could be independently controlled. A

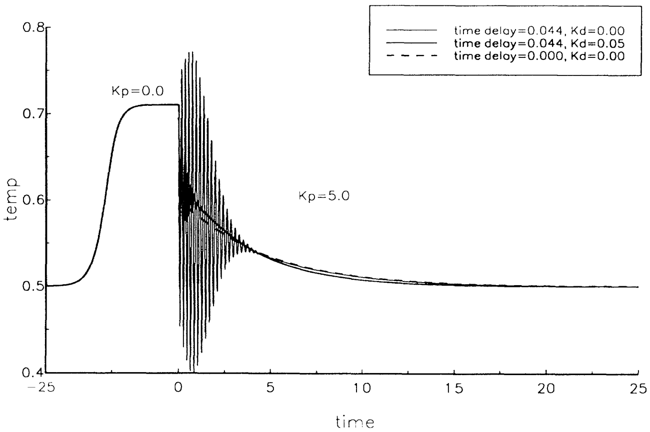

FIG. 3. The temperature $T(0,0,t)$ is depicted as a function of time for $R=3500$ ($r\sim 2$), $\mathrm{Pr}=0.02$ (i.e., mercury), and a cell of width 0.72. The simulation starts with a no-motion state and without a controller. The controller is switched on at time $t=0$. Three types of controls are examined: (i) proportional control (gain 5) without a time delay (dashed line); (ii) proportional control (gain 5) with a time delay of 0.045 (light solid line); and (iii) proportional (gain 5) and differential (gain 0.05) control with a time delay of 0.045 (heavy solid line).

representative result of one of our simulations is depicted in Fig. 3 for $R=3500$ ($r\sim 2$), $\mathrm{Pr}=0.02$ (i.e., mercury), and a cell of width 0.72. The critical Rayleigh numbers for the uncontrolled and controlled ($K=5$) systems with a similar cell width are about 1929 and 5480, respectively. Figure 3 depicts the temperature $T(0,0,t)$ as a function of time. The simulation starts with a no-motion state and without a controller. Since the Rayleigh number for the uncontrolled system is supercritical, an intentionally introduced disturbance induces counterclockwise flow in the cell. Time-independent flow is established once transients die out. Once convection has been established, the controller is switched on (at time $t=0$ in Fig. 3). Three types of controls are examined: (i) proportional control (gain 5) without a time delay (dashed line); (ii) proportional control (gain 5) with time delay of 0.045, which for a cell of 0.01 m height and an average temperature of 50 °C corresponds to 0.9 s (light solid line); and (iii) proportional (gain 5) and differential (gain 0.05) control with the same time delay (heavy solid line). Witness that in each of the three cases set forth above, the controller successfully suppresses the motion and the temperature decays to its conductive, no-motion value of 0.5. In the absence of a time delay, the decay is monotone. In the presence of a time delay, the decay is oscillatory. These oscillations can significantly be reduced by engaging a differential controller. In Fig. 3, we imposed on the controller the daunting task of suppressing already established motion. Had we gradually increased the Rayleigh number from its no-motion value while the controller was being engaged, the control task would have been much easier.

We have carried out similar simulations for other parameter values to find that the controller can cope with larger Rayleigh numbers and larger time delays than the one depicted in Fig. 3. Also, one can use more elaborate control strategies to improve the controller's performance.

Note that the magnitude of the bottom temperature modulations required to affect the control is proportional to the noise level in the controlled system. In the presence of low amplitude noise, the deviations of the bottom temperature from its nominal value will be small. In the presence of large amplitude noise the controller may saturate.

In summary, we have demonstrated theoretically that a simple control strategy can significantly shift the bifurcation point for the onset of convection. Behringer [1] reports observations of turbulent flow in an infinite layer at Rayleigh numbers as low as 2000 while we can maintain the no-motion state for $R>10^4$. Thus, the controller is clearly capable of suppressing turbulent behavior. Of course, the theoretical results need to be experimentally verified. In practice, the success of the controller will depend to a large extent on the magnitude of background noise in the system. Nevertheless, one ought to keep in mind that we have used here a very simple control strategy. It is more than likely that additional gains can be obtained by employing control strategies more sophisticated than the one described here. If that is the case, the proposed technique or modifications thereof may be useful in a variety of material processing and crystal growth processes. The current remedy for convective currents is to go to a low gravity environment out in space. We are proposing here a less glamorous but also considerably less expensive alternative.

This work was supported, in part, by the Department of Energy, Office of Basic Energy Sciences, through Grant No. DE-FG02-92ER14271. The numerical simulations were conducted using the Cornell National Supercomputer Facility. We are grateful to Professor E. Doedel for providing us with the software package AUTO.

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[1] For a lucid review of the Rayleigh-Bénard problem, see R. P. Behringer, Rev. Mod. Phys. **57**, 657 (1985).

[2] J. Singer, Y-Z. Wang, and H. H. Bau, Phys. Rev. Lett. **66**, 1123–1125 (1991); J. Singer and H. H. Bau, Phys. Fluids A **3**, 2859–2865 (1991); Y-Z. Wang, J. Singer, and H. H. Bau, J. Fluid Mech. **237**, 479–498 (1992).

[3] E. N. Lorenz, J. Atmosph. Sci. **20**, 130–141 (1963).

[4] S. Chandrasekhar, *Hydrodynamic and Hydromagnetic Stability* (Dover, New York, 1981), Chap. 2.

[5] W. H. Reid and D. L. Harris, Phys. Fluids **1**, 102–110 (1958).

[6] E. Doedel, H. B. Keller, and J. P. Kernevez, Int. J. Bifurcation Chaos **1**, 493–520 (1991); **1**, 745–772 (1991); E. Doedel and J. P. Kernevez, "AUTO: Software for Continuation and Bifurcation Problems in Ordinary Differential Equations," Applied Mathematics, California Institute of Technology, 1986 (unpublished).