Stress-Energy Tensor of Quantized Scalar Fields in Static Black Hole Spacetimes

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We present a method for the numerical computation of the stress-energy tensor of a quantized scalar field in a general static spherically symmetric spacetime, with or without horizon. The scalar field may have arbitrary curvature coupling and mass. Our method leads in a natural way to a new analytic approximation to the stress-energy tensor for massless scalar fields in these spacetimes. We use the results to compute stress-energy tensors of quantized scalar fields in Schwarzschild and Reissner-Nordstrøm black hole spacetimes.

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The discovery by Hawking [1] that black holes emit radiation in a thermal state showed that quantum fields have a profound effect on black hole spacetimes. This discovery placed black hole thermodynamics on a firm foundation; it also led to the conclusion that black holes in isolation (not surrounded by a heat bath) will evolve by evaporation, dwindling until they become objects of such size and mass that quantum gravity is required to adequately describe them.

Early calculations of black hole radiance proceeded by studying the scattering of waves (the modes of the quantized field) in the fixed background spacetime. This approach allows one to obtain information about particle production. However, it does not give information about vacuum polarization effects nor does it tell one how the spacetime geometry near a black hole is changed by the stress energy of the quantum fields. The latter is especially important because the thermodynamic properties and evolutionary history of a black hole are determined by its spacetime geometry.

One way to obtain more information about quantum effects is to compute the stress-energy tensor of the quantized field, $\langle T_{\mu\nu}\rangle$. Knowledge of $\langle T_{\mu\nu}\rangle$ gives information on vacuum polarization and particle production in a manner independent of any particular choice of mode decomposition, unlike, e.g., the particle number operator. Further, if $\langle T_{\mu\nu}\rangle$ can be computed for a general class of spacetimes then the semiclassical backreaction equations

$$G_{\mu\nu}=8\pi\langle T_{\mu\nu}\rangle \tag{1}$$

can be solved for that class of spacetimes.

In this Letter we present a numerical method which allows for the computation of $\langle T_{\mu\nu}\rangle$ for a quantized scalar field with arbitrary mass and curvature coupling in a general static spherically symmetric spacetime. Included in this class of spacetimes are the Schwarzschild and Reissner-Nordstrøm spacetimes which describe static uncharged and charged black holes, respectively. A similar method of calculating the vacuum polarization, $\langle\phi^2\rangle$, in a general static spherical spacetimes, was presented in Ref. [2]. The fact that $\langle T_{\mu\nu}\rangle$ can now be computed in these spacetimes will make it possible to obtain static spherically symmetric solutions to the semiclassical backreaction equations. Such solutions will give substantial insight into the question of how quantum effects distort the spacetime geometry near a static black hole which is in thermal equilibrium with its surroundings. They will also provide self-consistent equilibria for the study of black hole thermodynamics.

Because of the difficulty involved in numerically computing $\langle T_{\mu\nu}\rangle$ it is often useful to approximate it analytically. For conformally coupled massless scalar fields, such approximations have previously been derived by Page [3] for Schwarzschild spacetime and by Frolov and Zel'nikov [4] for a general static spacetime. We present here an analytic approximation for $\langle T_{\mu\nu}\rangle$ for a massless scalar field with arbitrary curvature coupling in a general static spherically symmetric spacetime. This new approximation is obtained in a natural fashion from our approach to the full numerical calculation of $\langle T_{\mu\nu}\rangle$. For the particular case of conformal coupling the approximation reduces to that of Frolov and Zel'nikov. It is therefore equivalent to Page's approximation for a conformally coupled massless scalar field in Schwarzschild spacetime. The derivation of our approximation is based solely on quantum field theory. The original derivation of Frolov and Zel'nikov's approximation was based on geometrical concerns. Our derivation provides the first justification for their approximation from the viewpoint of quantum field theory.

We have used our numerical method to compute the stress-energy tensor for quantized scalar fields in the Hartle-Hawking state in Schwarzschild and Reissner-Nordstrøm black hole spacetimes. Computations have been carried out for both the massless and massive scalar field with arbitrary curvature couplings. In this Letter we present results for massless fields; our results for massive fields will be presented elsewhere. These are the first numerical calculations of $\langle T_{\mu\nu}\rangle$ for massive scalar fields and for nonconformally coupled massless scalar fields in Schwarzschild spacetime. Previous calculations of $\langle T_{\mu\nu}\rangle$ for scalar fields in the Schwarzschild black hole spacetime by Fawcett [5] and by Howard and Candelas [6] have been limited to the special case of a massless conformally coupled field. The vacuum stress energy has also been computed for the electromagnetic field in Schwarzschild spacetime by Jensen and Ottewill [7]. No numerical computations of the vacuum stress-energy tensor have previously been done for any quantized fields in Reissner-Nordstrøm spacetimes.

The rest of this Letter proceeds as follows: We begin by deriving an exact expression for the stress-energy tensor of a quantized scalar field with arbitrary mass and curvature coupling in a general static spherically symmetric spacetime. Both the zero and nonzero temperature cases are considered. The resulting expression consists of two parts: an analytic portion which becomes our new approximation, and a mode sum which must be computed numerically. We next examine and compare the numerical and approximate expressions for $\langle T_{\mu\nu}\rangle$ for massless fields in the Schwarzschild and Reissner-Nordstrøm spacetimes. Complete details of the derivation of both the exact and approximate expressions for the stress-energy tensor, a similar approximation scheme for the vacuum polarization, $\langle\phi^2\rangle$, and numerical results for massive scalar fields will appear in a separate paper.

The numerical computation of the quantum stress-energy tensor proceeds in much the same manner as that of Howard and Candelas' [6] calculation of $\langle T_{\mu\nu}\rangle$ for massless conformally coupled scalar fields in Schwarzschild spacetime. As in their calculation, a Euclidean space approach is used. The metric for a general static spherically symmetric spacetime when analytically continued into Euclidean space is [8]

$$ds^2=f(r)d\tau^2+h(r)dr^2+r^2d\theta^2+r^2\sin^2\theta d\phi^2 . \quad (2)$$

Here f and h are arbitrary functions of r which, if the space is asymptotically flat, become constant in the limit as r approaches infinity.

$\langle T_{\mu\nu}\rangle$ is computed using the method of point splitting [9]. One begins by noting that $\langle T_{\mu\nu}\rangle$ can be obtained by taking derivatives of the quantity $\langle\phi(x)\phi(x')\rangle$ and then letting $x'\to x$. The calculation is simplified by noting that in the limit $x'\to x$, $\langle\phi(x)\phi(x')\rangle=G_E(x,x')$, where G_E is the Euclidean space Green function. In Ref. [2] it was shown that for the metric of Eq. (1) $G_E(x,x')$ is given by

$$G_E(x,x')=\int d\mu\, e^{i\omega(\tau-\tau')}\sum_{l=0}^{\infty}(2l+1)P_l(\cos\gamma)C_{\omega l}p_{\omega l}(r_<)q_{\omega l}(r_>) , \quad (3)$$

where

$$\int d\mu\equiv\begin{cases}\dfrac{1}{8\pi^2}\displaystyle\int d\omega, & T=0,\\ \dfrac{T}{4\pi}\displaystyle\sum_{n=-\infty}^{\infty}, & T>0.\end{cases}$$

Here P_l is a Legendre polynomial, $\cos\gamma\equiv\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos(\phi-\phi')$, $C_{\omega l}$ is a normalization constant, $r_>$ ($r_<$) is the greater (lesser) of r and r', T is the temperature of the field, and $\omega=2\pi nT$ if $T\neq0$. The modes $p_{\omega l}$ and $q_{\omega l}$ obey the equation

$$\frac{1}{h}\frac{d^2S}{dr^2}+\left[\frac{2}{rh}+\frac{1}{2fh}\frac{df}{dr}+\frac{1}{2h^2}\frac{dh}{dr}\right]\frac{dS}{dr}-\left[\frac{\omega^2}{f}+\frac{l(l+1)}{r^2}+m^2+\xi R\right]S=0 . \quad (4)$$

The vacuum expectation value of the stress-energy tensor is given by the equation

$$\langle T_{\mu\nu}\rangle=\lim_{x'\to x}[(\tfrac{1}{2}-\xi)(g_\mu{}^{\alpha'}G_{E;\alpha'\nu}+g_\nu{}^{\alpha'}G_{E;\mu\alpha'})+2(\xi-\tfrac{1}{4})g_{\mu\nu}G_{E;\sigma}{}^{\sigma'}-\xi(G_{E;\mu\nu}+g_\mu{}^{\alpha'}g_\nu{}^{\beta'}G_{E;\alpha'\beta'})+2\xi g_{\mu\nu}(m^2+\xi R)G_E$$
$$+\xi(R_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}R)G_E-\tfrac{1}{2}m^2g_{\mu\nu}G_E] . \quad (5)$$

Substituting Eq. (3) into Eq. (5) results in an unrenormalized expression for $\langle T_{\mu\nu}\rangle$. This expression is renormalized by subtracting off the point-splitting counterterms obtained by Christensen using the DeWitt-Schwinger expansion [9]. For computational purposes we find it helpful to write the renormalized expression for $\langle T_{\mu\nu}\rangle$ in the following schematic form:

$$\langle T_{\mu\nu}\rangle_{\rm ren}=\lim_{x'\to x}(\langle T_{\mu\nu}\rangle_{\rm unren}-\langle T_{\mu\nu}\rangle_{\rm WKB})+(\langle T_{\mu\nu}\rangle_{\rm WKB}-\langle T_{\mu\nu}\rangle_{\rm WKBdiv})+(\langle T_{\mu\nu}\rangle_{\rm WKBdiv}-\langle T_{\mu\nu}\rangle_{\rm DS})$$
$$=\langle T_{\mu\nu}\rangle_{\rm modes}+\langle T_{\mu\nu}\rangle_{\rm WKBfin}+\langle T_{\mu\nu}\rangle_{\rm analytic} . \quad (6)$$

The procedure we use to compute the various terms in Eq. (6) is to split the points along the t direction and define $\epsilon=t-t'$. Then $\langle T_{\mu\nu}\rangle_{\rm unren}$ is computed using numerical solutions to the mode equation (4). $\langle T_{\mu\nu}\rangle_{\rm WKB}$ is computed by using the WKB approximation for the modes. This approximation has been outlined in detail in Ref. [2]. If at least a fourth order WKB expansion is used for the modes, then $\langle T_{\mu\nu}\rangle_{\rm modes}$ consists of finite sums and integrals over the modes. The higher the order used, the more rapidly the sums and integrals converge. We use a sixth order expansion for massive fields and an eighth order expansion for massless fields.

$\langle T_{\mu\nu}\rangle_{\rm WKBdiv}$ is computed by first computing the sums over l in $\langle T_{\mu\nu}\rangle_{\rm WKB}$ in the large ω limit. This results in an asymptotic expansion of $\langle T_{\mu\nu}\rangle_{\rm WKB}$ in inverse powers of ω. The expansion is truncated at order ω^{-1}. The sums or integrals over ω include $\omega=0$. Thus it is necessary to impose an infrared cutoff on those sums or integrals containing terms which are of order ω^{-1}. Since $\langle T_{\mu\nu}\rangle_{\rm WKBdiv}$ is both added and subtracted in Eq. (6), it is clear the $\langle T_{\mu\nu}\rangle_{\rm ren}$ is independent of the value of this cutoff. All of the ultraviolet divergences in $\langle T_{\mu\nu}\rangle_{\rm WKB}$ are contained in $\langle T_{\mu\nu}\rangle_{\rm WKBdiv}$. We compute $\langle T_{\mu\nu}\rangle_{\rm WKBfin}$ numerically because the sums and integrals are too complicated to be analytically computable.

$\langle T_{\mu\nu}\rangle_{\rm analytic}$ is computed by first computing the sums or integrals over ω in $\langle T_{\mu\nu}\rangle_{\rm WKBdiv}$ with $\epsilon\neq 0$. Then $g_{\alpha\beta}$ is expanded in powers of ϵ as are the terms in the point-splitting counterterms $\langle T_{\mu\nu}\rangle_{\rm DS}$. The difference is computed and then the limit $\epsilon\to 0$ is taken. Howard and Candelas [6] and Jensen and Ottewill [7] have derived analytic contributions to $\langle T_{\mu\nu}\rangle_{\rm ren}$ in this way for the conformally invariant scalar field and the electromagnetic field, respectively, in Schwarzschild spacetime.

The result contains a logarithmic term originating from $\langle T_{\mu\nu}\rangle_{\rm DS}$. For a massless scalar field this term contains an arbitrary constant [9]. This constant represents an ambiguity in the way in which the limit $m\to 0$ is computed for the renormalization counterterms. Such a term always appears for a massless field. Its existence is not a problem because its coefficient is proportional to the variation of the combination of a Weyl tensor squared term and a scalar curvature squared term in the gravitational Lagrangian. Thus a particular choice of value for the arbitrary constant corresponds to a finite renormalization of the coefficients of these terms in the gravitational Lagrangian. As such, the value of this constant must be fixed by experiment or observation.

For a massless scalar field, $\langle T_{\mu\nu}\rangle_{\rm analytic}$ can be used by itself as an approximation for $\langle T_{\mu\nu}\rangle_{\rm ren}$. However, for it to be a useful approximation the dependence on the infrared cutoff in $\langle T_{\mu\nu}\rangle_{\rm WKBdiv}$ must first be removed. We accomplish this by absorbing the cutoff into the definition of the arbitrary constant mentioned above. The result is an approximate stress-energy tensor which is conserved and which, for the conformally invariant scalar field, has a trace equal to the trace anomaly.

We find that for the special case of conformal coupling, $\xi=\frac{1}{6}$, $\langle T_{\mu\nu}\rangle_{\rm analytic}$ is equivalent to the approximation of Frolov and Zel'nikov [4] if the arbitrary constants $q_2^{(0)}$ and $q_1^{(2)}$ in their expression for $\langle T_{\mu\nu}\rangle$ are set equal to zero. Their arbitrary constant $q_1^{(0)}$ is related to the arbitrary constant discussed above. As a result, our approximation duplicates (for conformal coupling) Huang's results for the Frolov-Zel'nikov approximation in Reissner-Nordstrøm spacetimes [10] and is also equivalent to Page's approximation [3] for the stress-energy tensor of a conformally coupled scalar field in any static spherically symmetric Einstein spacetime.

Expressions for the components of $\langle T_{\mu\nu}\rangle_{\rm analytic}$ in a general static spherically symmetric spacetime are too long to be shown here. In a Reissner-Nordstrøm spacetime, the expressions simplify considerably. We display one component of $\langle T_{\mu\nu}\rangle_{\rm analytic}$ below. Before doing so, it is useful to note that for Reissner-Nordstrøm spacetimes, the scalar curvature R is zero. Examination of Eqs. (3)-(5) shows that, in this case, the stress-energy tensor (exact or approximate) is a linear function of the curvature coupling ξ. It can thus be written in the following form:

$$\langle T_{\mu\nu}\rangle = C_{\mu\nu} + (\xi - \tfrac{1}{6}) D_{\mu\nu}. \tag{7}$$

For a Reissner-Nordstrøm black hole of mass M and charge Q we find

$$\begin{aligned}(C_\theta{}^\theta)_a = {} & (2880\pi^2 r^8\Delta^2)^{-1}[2\kappa^4 r^{12} - 32Q^2r^6 - 8M^2r^6 + 244MQ^2r^5 + 24M^3r^5 - 141Q^4r^4 - 580M^2Q^2r^4 - 18M^4r^4 \\ & + 636MQ^4r^3 + 440M^3Q^2r^3 - 174Q^6r^2 - 700M^2Q^4r^2 + 376MQ^6r - 67Q^8] \\ & - \frac{Q^2\Delta}{60\pi^2 r^8}\left[C + \frac{1}{2}\ln\left(\frac{\Delta\mu^2}{4r^2}\right)\right],\end{aligned} \tag{8}$$

$$\begin{aligned}(D_\theta{}^\theta)_a = {} & (48\pi^2 r^6\Delta^2)^{-1}(2\kappa^2 M r^7 - 3\kappa^2Q^2r^6 + \kappa^2Q^4r^4 + 16Q^2r^4 - 16M^2r^4 - 54MQ^2r^3 + 54M^3r^3 + 21Q^4r^2 \\ & + 27M^2Q^2r^2 - 48M^4r^2 - 40MQ^4r + 40M^3Q^2r + 9Q^6 - 9M^2Q^4),\end{aligned} \tag{9}$$

where $\Delta = r^2 - 2Mr + Q^2$, κ is the surface gravity of the black hole, C is Euler's constant, and μ is the arbitrary constant discussed above.

Frolov and Zel'nikov [4] have shown that their approximate stress-energy tensor for a conformally invariant field diverges on the event horizon of a static charged black hole. The divergence occurs regardless of how small the charge is, so long as it is nonzero. Our more general analytical approximation shows that this divergence also occurs for non-conformally coupled massless scalar fields, independent of the value of ξ. Whether this divergence is real or simply an

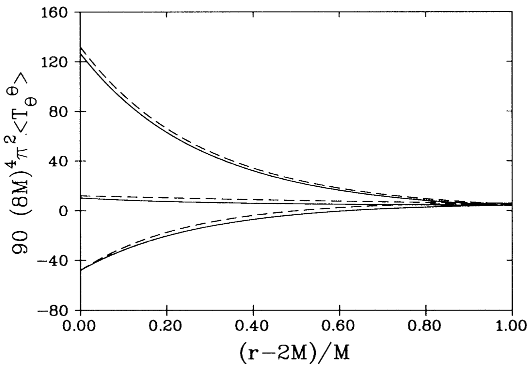

FIG. 1. The curves in this figure display the values of $\langle T_\theta{}^\theta\rangle$ for a quantized scalar field around a Schwarzschild black hole with $\xi=0,\frac{1}{6},\frac{1}{4}$ from top to bottom at the event horizon, $r=2M$. The solid curves are the results of our numerical calculations; the dashed curves show the analytical approximation.

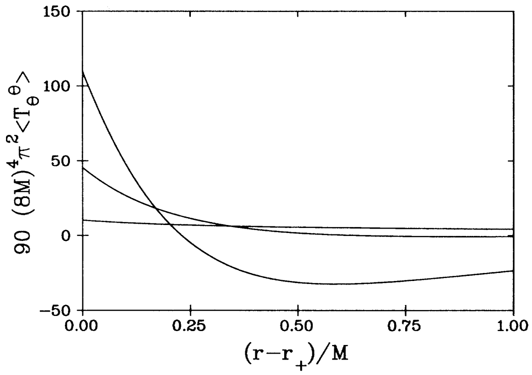

FIG. 2. The curves in this figure display the values of $\langle T_\theta{}^\theta\rangle$ for a conformally coupled field around a Reissner-Nordstrøm black hole with $|Q|/M=0.99,0.8,0$ from top to bottom at the event horizon, $r=r_+$.

artifact of the approximation is as yet unknown. Our numerical results appear to indicate that the divergence is an artifact of the approximation, but it is difficult to be certain since we cannot rigorously demonstrate that the quantity $(\langle T_r{}^r\rangle-\langle T_t{}^t\rangle)/(r-r_+)$ is finite in the limit that $r\to r_+$. Numerically calculated values for the stress tensor components, accurately known to only a finite number of digits, are never sufficient to prove the regularity of the stress-energy tensor on the horizon.

We have numerically computed the stress-energy tensor for quantized scalar fields in Schwarzschild and Reissner-Nordstrøm spacetimes. Some of our results for massless scalar fields are shown in Figs. 1 and 2. We have set the arbitrary constant μ equal to 1. In Fig. 1 we show $\langle T_\theta{}^\theta\rangle$ for quantized scalar fields in the Schwarzschild spacetime with $\xi=0,\frac{1}{6},\frac{1}{4}$. The solid lines are the full numerical values, while the dashed lines are the new approximation. In Fig. 2 we show $\langle T_\theta{}^\theta\rangle$ for a conformal scalar field in the Reissner-Nordstrøm spacetimes with $|Q|/M=0,0.8,0.99$, where Q is the charge of the black hole and M is its mass. The event horizon is at $r=r_+=M+(M^2-Q^2)^{1/2}$.

As can be seen from Fig. 1, the accuracy of the analytical approximation does not depend strongly upon the value of ξ. We find that the analytical approximation works best for small values of $|Q|/M$. It works rather poorly near the event horizon for $|Q|/M\approx 1$.

The numerically calculated value of $\langle T_\theta{}^\theta\rangle$ for the conformally coupled field at the event horizon rises as $|Q|/M$ is increased, reaches a finite maximum when $|Q|/M\simeq 0.98$, and then decreases as the extremal case $(Q^2=M^2)$ is approached. Similar behavior was found for the vacuum polarization $\langle\phi^2\rangle$ [2].

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