# Intermittency and Predictability in Turbulence

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We discuss the relation between predictability and the sensitive dependence on initial conditions in turbulent flows. We find that the maximum Lyapunov exponent $\lambda$ and the variance of the effective Lyapunov exponent diverge as a power of the Reynolds number with scaling exponents which can be calculated from the eddy turnover time at the Kolmogorov length using the multifractal approach. The intermittency leads to long tails in the distribution of the predictability time. The typical predictability time $T$ is related to the value of $\lambda^{-1}$. We provide numerical evidence of this picture within the framework of a cascade model for three-dimensional fully developed turbulence.

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The intermittency of the energy dissipation in three-dimensional (3D) turbulence causes the failure of the Kolmogorov K41 theory which is based on dimensional arguments [1]. The most evident signature of this failure is the appearance of a correction to the scaling exponents $\zeta_p$ of the velocity structure functions $\langle|\mathbf{u}(\mathbf{x}+\mathbf{r})-\mathbf{u}(\mathbf{x})|^p\rangle \sim r^{\zeta_p}$, where $\langle\ \rangle$ indicates a spatial or temporal average; in the K41 theory $\zeta_p=p/3$ [2]. The experimentally observed nonlinear shape of $\zeta_p$ is well described by the multifractal formalism and reproduced by multiplicative random processes such as the random beta model [3].

The word intermittency is often used with different meanings. In this Letter intermittency of the energy dissipation indicates the presence of spatial and temporal bursts in the velocity gradients, and dynamical intermittency indicates the existence of temporal fluctuations in the degree of chaos.

The dynamical intermittency may have rather important effects on the predictability in turbulent flows, a phenomenon which has not been fully investigated. The purpose of this Letter is to discuss the consequences of the multifractality in energy dissipation on the growth of a disturbance on the velocity field $\mathbf{u}(\mathbf{x})$ and on the statistics of the predictability time in turbulent flows.

The chaotic behavior of a dynamical system is characterized by a positive maximum Lyapunov exponent $\lambda$, which measures the typical exponential growth rate of an infinitesimal disturbance [4]. In 3D fully developed turbulence, the maximum Lyapunov exponent should be roughly proportional to the inverse of the smallest characteristic time of the system, that is, the turnover time $\tau$ of eddies of the size of the Kolmogorov length $\eta$ (the viscous cutoff). By dimensional counting, after introducing the adimensional scaling parameter $l=r/L$, the turnover time of an eddy of size $l$ is $\tau(l)\sim T_0 l^{1-h}$, where $h$ is the Hoelder exponent of the velocity difference in the eddy $|\mathbf{u}(\mathbf{x}+\mathbf{r})-\mathbf{u}(\mathbf{x})|\sim Vl^h$. In these relations, we have used the typical large length scale of the system $L$, the corresponding typical speed $V=(\epsilon L)^{1/3}$, and time $T_0=L/V=(L^2/\epsilon)^{1/3}$ which are expressed as usual in terms of the spatial average of the energy dissipation density $\epsilon=V^3/L$. Moreover, it can be shown that the viscous cutoff vanishes as a power of the Reynolds number $\mathrm{Re}=VL/\nu$ ($\nu$ is the viscosity), i.e., $\eta\sim L\,\mathrm{Re}^{-1/(1+h)}$ [5]. These dimensional relations imply that the maximum Lyapunov exponent should scale as

$$\lambda\sim\frac{1}{\tau(\eta)}\sim\frac{1}{T_0}\mathrm{Re}^{\alpha} \tag{1}$$

with

$$\alpha=\frac{1-h}{1+h}.$$

In the K41 theory $h=\frac{1}{3}$ for all space points so that $\alpha=\frac{1}{2}$, as first pointed out by Ruelle [6].

However, one expects that the presence of quiescent quasilaminar periods should change the chaotic features of the fluid flow. In fact, the intermittency of energy dissipation can be described by introducing a spectrum of scaling exponents $h$. In the multifractal approach, the probability that the velocity difference scales as $|\mathbf{u}(\mathbf{x}+\mathbf{r})-\mathbf{u}(\mathbf{x})|\sim Vl^h$ is assumed to be $P_l(h)\sim l^{3-D(h)}$, where the function $D(h)$ is given by the Legendre transform $\zeta_p=\min_h[hp-D(h)+3]$. The multifractality also implies the existence of a spectrum of viscous cutoffs, since each $h$ selects a different damping scale $\eta(h)\sim L\times\mathrm{Re}^{-1/(1+h)}$, and hence a spectrum of turnover times. To find the Lyapunov exponent, we have to integrate over the $h$ distribution $P_l(h)$ at scale $l=\eta(h)/L$:

$$\lambda\sim\int\tau(h)^{-1}P_l(h)dh\sim\frac{1}{T_0}\int\left(\frac{\eta}{L}\right)^{h-D(h)+2}dh\,, \tag{2}$$

where $\tau(h)$ is the turnover time of an eddy of scale $\eta(h)$, so that $\tau(h)/T_0\sim[\eta(h)/L]^{1-h}\sim\mathrm{Re}^{-(1-h)/(1+h)}$. In the limit $\mathrm{Re}\to\infty$ the viscous cutoffs $\eta(h)$ vanish and the integral can be estimated by the saddle point method,

$$\lambda \sim (1/T_0)\mathrm{Re}^{\alpha} \tag{3}$$

with

$$\alpha = \max_h \frac{D(h)-2-h}{1+h} .$$

The value of $\alpha$ depends on $D(h)$. By using the function $D(h)$ obtained from fitting the exponents $\zeta_q$ with the random beta model [3], we find $\alpha = 0.459\ldots$, which is smaller than the Ruelle prediction [6] $\alpha = 0.5$.

The finite-time fluctuation of the degree of chaos can be characterized by an effective Lyapunov exponent $\gamma$ [7,8], defined as the exponential divergence rate after a time delay $\tau$ of two trajectories close at time $t$, i.e.,

$$\gamma_\tau(t) = \frac{1}{\tau} \ln \frac{||\delta \mathbf{u}(t+\tau)||}{||\delta \mathbf{u}(t)||} , \tag{4}$$

where $\delta \mathbf{u}$ is the infinitesimal difference between two velocity fields evolving under the same equation (i.e., the Navier-Stokes equations).

The Lyapunov exponent is given by $\lambda = \lim_{\tau \to \infty} \gamma_\tau$ which has the same value for almost all initial conditions. However, there are fluctuations for finite $\tau$ and, in general cases, the probability of finding $\gamma_\tau \neq \lambda$ scales as $P(\gamma) \sim \exp[-S(\gamma)\tau]$, where $S(\gamma) \geq 0$ with the equal sign for $\gamma = \lambda$. Usually, for small perturbations, i.e., when $\gamma - \lambda = O(\tau^{-1/2})$, central limit arguments can be applied so that $S(\gamma)$ is quadratic in $\gamma - \lambda$. However, the Gaussian approximation does not hold for large fluctuations and $S(\gamma)$ is an important characterization of a dynamical system [7,8]. It is worth stressing that the Gaussian approximation [that is, a parabolic shape of $S(\gamma)$ around its minimum] can fail even for small $|\gamma - \lambda|$, and that there are cases where the function $S(\gamma)$ is not well defined; see Ref. [9].

The parameters $\lambda$ and $\mu$ give the main characterization of the $\gamma$ distribution. It has been shown that $\mu/\lambda = 1$ separates weak from strong intermittency (for a detailed discussion on this point see Ref. [7]).

In order to test our arguments, we have numerically studied a shell model [10,11] for the energy cascade in fully developed turbulence. The model is an approximation of the Navier-Stokes equations obtained by dividing the Fourier space into shells of wave numbers $k_n < |\mathbf{k}| < k_{n+1}$. A complex scalar $u_n$ is associated with the $n$th shell individuated by $k_n = k_0 2^n$. It represents the Fourier transform of the velocity field integrated over the shell volume. Since the energy cascade in turbulence is believed to be local in the $k$ space with an exponentially decreasing interaction among shells, it is reasonable to consider only the interactions of a shell with its nearest and next-nearest neighbors. The Navier-Stokes equations are then approximated by a set of ordinary differential equations. In this dynamical approach one uses an ergodic hypothesis so that the average $\langle \ \rangle$ is a time average.

The shell model exhibits exponents $\zeta_p$ that are nonlinear in $p$, in agreement with experimental data [12]. These corrections to the Kolmogorov theory are due to the intermittency in the dynamical evolution, as energy bursts are observed to interrupt quiescent laminar periods when there is a sudden increase of the effective Lyapunov experiment.

The Reynolds number is obtained from numerical calculations for a shell model with $N = 27$ shells. In our simulations we change Re by changing only the value of the viscosity $\nu$. The correction to the Ruelle prediction [6] $\lambda \sim \mathrm{Re}^{1/2}$ is clearly evident and agrees with Eq. (3). We have also computed the variance of the finite-time fluctuations as a function of Re. Figure 1 shows that it diverges as $\mu(\mathrm{Re}) \sim \mathrm{Re}^{w}$ with $w \simeq 0.8$. Although it is sensible to expect $w > 0$ in real turbulent fluid, we cannot exclude that $w \simeq 0.8$ is due to the particular form of the time correlations in the shell model. In fact, $w$ is related to the decay rate of time correlations. The variance of the fluctuations of the local Lyapunov exponents can be computed from the multifractal spectrum of $\tau(h)^{-2}$. Noting that $\gamma_\tau(t) = (1/\tau)\int_t^{t+\tau} \gamma_0(t')dt'$, an explicit calculation leads to

$$\mu \sim \int_0^\infty \langle [\gamma_0(t+t') - \lambda][\gamma_0(t) - \lambda] \rangle dt'$$
$$\sim \langle (\gamma_0 - \lambda)^2 \rangle \int_0^\infty C(t')dt' ,$$

where $C(t')$ is the normalized correlation function of the effective Lyapunov exponents $\langle [\gamma_0(t) - \lambda][\gamma_0(t+t') - \lambda] \rangle / \langle [\gamma_0(t) - \lambda]^2 \rangle$, which has the same qualitative behavior of the energy dissipation correlation function. Thus we define the characteristic time

$$t_c = \int_0^\infty C(t')dt' \sim T_0 \mathrm{Re}^{-z} , \tag{5}$$

which is assumed to vanish as a power of Re. Moreover, $\langle [\gamma_0(t) - \lambda]^2 \rangle$ can be estimated by repeating the argu-

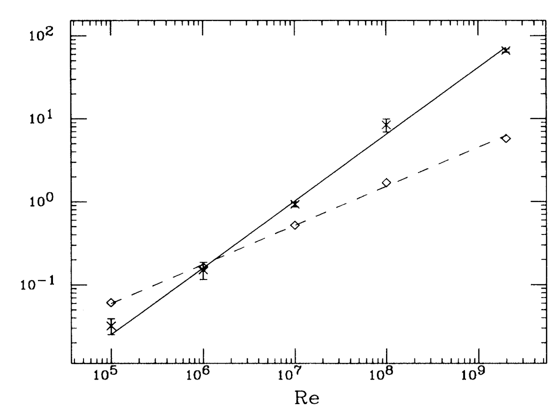


FIG. 1. The Lyapunov exponent $\lambda$ (diamonds) and the variance $\mu$ (crosses) as a function of the Reynolds numbers from a shell model calculation with $N = 27$ shells. The dashed line is the multifractal prediction $\lambda \sim \mathrm{Re}^{\alpha}$ with $\alpha = 0.459$, where the function $D(h)$ is given by the random beta model fit of the $\zeta_p$ exponents [3]. The full line indicates $\mu \sim \mathrm{Re}^{w}$ with $w = 0.8$.

ments used for $\lambda$, so that

$$\langle \gamma_0^2 \rangle \sim \int \tau(h)^{-2} P_\eta(h) dh \sim \frac{1}{T_0^2} \mathrm{Re}^y$$

with

$$y = \max_h \frac{D(h) - 1 - 2h}{1+h} = 1 .$$

The result $y=1$ is model independent, since $\langle \gamma_0^2 \rangle \sim \mathrm{Re}\bar{\epsilon}$, where the spatial average of the energy dissipation density $\bar{\epsilon}$ is a finite quantity independent of Re. The fact that $\langle \gamma_0^2 \rangle \gg \lambda^2$ at high Re implies

$$\mu \sim \langle \gamma_0^2 \rangle t_c \simeq (1/T_0) \mathrm{Re}^w$$

with

$$w = 1 - z .$$

We have performed numerical estimates of the correlation decay times $t_c$ in the shell model and found that this relation is satisfied with $z \simeq 0.2$.

Let us note that in the absence of intermittency one may expect that $t_c \sim \lambda^{-1}$, and thus $z \simeq \frac{1}{2}$. The fact that $z \simeq 0.2$ indicates that the presence of quiescent periods in the turbulent activity is much more relevant for the decay rate of time correlations than for the Lyapunov exponent.

The basic qualitative feature of our results is just the dynamical counterpart of multifractality of energy dissipation in 3D space. In generic chaotic systems a lower bound of $t_c$ is given by $\lambda^{-1}$. It follows that $w \geq \frac{1}{2}$ and $w > \alpha$, implying that $\mu/\lambda$ diverges as $\mathrm{Re} \to \infty$, and so the dynamical intermittency.

Another important signature of intermittency appears in the statistics of the predictability time $T$, defined as the time needed for a disturbance $\delta$ on the velocity field, localized at the Kolmogorov scale $\eta(h=\frac{1}{3})$, to affect the large scales. In practice we compute the time necessary for a small error on the dissipative shell to become larger than a given threshold value $g$ in one of the first shells. The toleration parameter $g$ is the maximal allowed ignorance on the system state in the large length structures (the energy containing eddies).

Numerically, the time $T$ is observed not to be constant but to be strongly dependent on the degree of chaos: If the system undergoes an energy burst, the predictability time is very small. On the other hand, if the system is in a laminar period, the predictability time can be very large. Figure 2 shows the probability distribution function (PDF) of $T$ for two different values of Re. At $\mathrm{Re} \approx 10^6$ we observe a rather peaked PDF with an almost Gaussian shape. For larger values of Re ($\mathrm{Re} \approx 2 \times 10^9$) the distribution gets an exponential tail, indicating the possibility of large excursions in the value of $T$, depending on whether the system is in a turbulent or in a purely laminar period. Furthermore, the typical predictability time $T_t$ (the $T$ value where the PDF reaches its maximum) is very dependent on the Lyapunov exponent and hence on the Reynolds number. In the shell model, one has roughly $T_t \sim 1/\lambda$, so that the typical predictability time decreases as a power of Re. Moreover, at increasing Re the occurrence of large values of $(T-T_t)/T_t$ is more and more likely. The above scenario does not depend on the values of the threshold $g$. Our observations are quite different from some previous results, suggesting that the predictability time is proportional to $T_0$, the turnover time of the energy containing eddies, and hence independent of Re [13]. The gross features of the probability distributions shown in Fig. 2 do not depend on the particular dynamical system considered but only on the degree of intermittency measured by $\mu/\lambda$: When $\mu/\lambda \gg 1$ the probability distribution of the predictability time has long exponential tails, while for $\mu/\lambda \leq 1$ it is very peaked. For instance, the long exponential tail appears in the Lorentz model with $r$ slightly larger than $r_c \simeq 166.07$ or in the Pomeau-Manneville map, near the intermittent transition, as $\mu/\lambda$ increases. Therefore, we can safely say that the mechanism for the occurrence of exponential tails is not an artifact of the shell model, but a rather robust feature of highly intermittent systems.

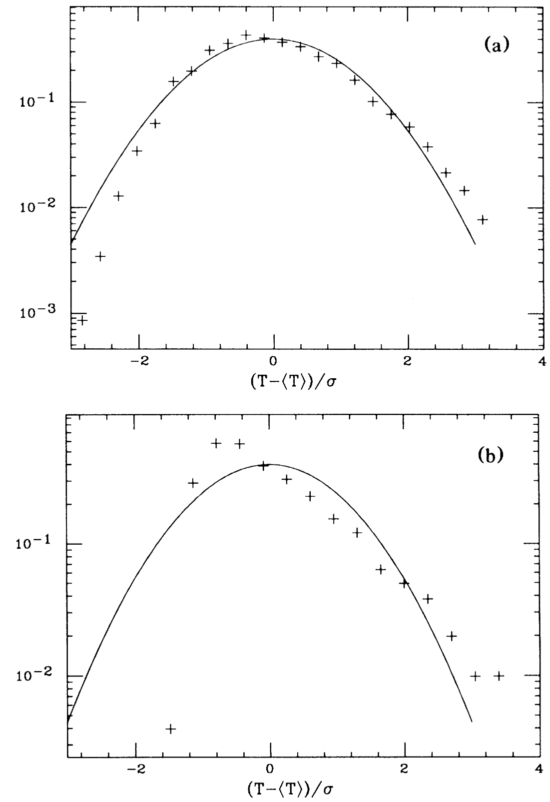


FIG. 2. Rescaled probability distribution functions (PDF) of the predictability time $T$: $\sigma P(T)$ vs $(T - \langle T \rangle)/\sigma$ for (a) $\mathrm{Re} = 10^6$ and (b) $\mathrm{Re} = 2 \times 10^9$. The respective average values are $\langle T \rangle = 84.0$, 6.32 and the standard deviations $\sigma = \langle (T - \langle T \rangle)^2 \rangle^{1/2}$ are 22.2 and 3.16. The full line is the standard Gaussian.

In conclusion, we have shown that the multifractal description of the energy dissipation in turbulence should have rather important implications for the chaotic behavior of turbulent fluids. In particular, we have found the scaling of the Lyapunov exponent with the Reynolds number in terms of the multifractal spectrum. Our calculations predict that multifractality gives rise to a strongly intermittent chaotic regime at high Reynolds number. In fact, the variance of the fluctuations of the effective Lyapunov exponent is found to diverge with the Reynolds number in a shell model for turbulence. We thus argue that the mean predictability time depends on the Reynolds number (it vanishes as $\mathrm{Re}^{-\alpha}$ with $\alpha \simeq 0.46$) and that the occurrence of long predictability times becomes more probable at increasing Re. In this sense, fully developed turbulence exhibits a smooth transition from a quasi Gaussian (weak turbulence) toward an intermittent regime (strong turbulence) for the statistics of the predictability time.

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[1] A. N. Kolmogorov, C. R. (Dokl.) Acad. Sci. USSR **30**, 301 (1941).

[2] A. S. Monin and A. M. Yaglom, *Statistical Fluid Mechanics* (MIT, Cambridge, MA, 1975), Vol. II.

[3] R. Benzi, G. Paladin, G. Parisi, and A. Vulpiani, J. Phys. A **17**, 3521 (1984); G. Parisi and U. Frisch, in *Turbulence and Predictability of Geophysical Flows and Climatic Dynamics,* edited by M. Ghil *et al.* (North-Holland, New York, 1985), p. 84; G. Paladin and A. Vulpiani, Phys. Rep. **156**, 147 (1987).

[4] G. Benettin, L. Galgani, and J. M. Strelcyn, Phys. Rev. A **14**, 2338 (1976).

[5] G. Paladin and A. Vulpiani, Phys. Rev. A **35**, 1971 (1987). For an important physical consequence of the existence of a spectrum of viscous cutoffs see U. Frisch and M. Vergassola, Europhys. Lett. **14**, 439 (1991).

[6] D. Ruelle, Phys. Lett. **72A**, 81 (1979).

[7] R. Benzi, G. Paladin, G. Parisi, and A. Vulpiani, J. Phys. A **18**, 2157 (1985).

[8] J. P. Eckmann and I. Procaccia, Phys. Rev. A **34**, 659 (1986); G. Paladin, L. Peliti, and A. Vulpiani, J. Phys. A **19**, L991 (1986).

[9] P. Grasberger, R. Badii, and A. Politi, J. Stat. Phys. **51**, 135 (1988). In the context of an earthquake model, we have found that the $\mu$ and $S(\gamma)$ are not well defined (they diverge as $\tau \rightarrow \infty$) though $\lambda$ is a finite positive quantity [A. Crisanti, M. H. Jensen, A. Vulpiani, and G. Paladin, Phys. Rev. A **46**, 7363 (1992)].

[10] M. Yamada and K. Ohkitani, J. Phys. Soc. Jpn. **56**, 4210 (1987); Prog. Theor. Phys. **79**, 1265 (1988).

[11] M. H. Jensen, G. Paladin, and A. Vulpiani, Phys. Rev. A **43**, 798 (1991).

[12] F. Anselmet, Y. Gagne, E. J. Hopfinger, and R. Antonia, J. Fluid Mech. **140**, 63 (1984).

[13] C. E. Leith and R. H. Kraichnan, J. Atmos. Sci. **29**, 1041 (1972); D. K. Lilly, in *Dynamical Meteorology,* edited by P. Morel (D. Reidel, Boston, 1973), p. 353.