Vector Order Parameter for an Unpolarized Laser and its Vectorial Topological Defects

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We consider the full set of equations ruling the interaction of an electromagnetic field with matter in a laser, without assuming that the direction of the transverse electric field is fixed. Near the lasing threshold, we reduce the dynamics to its normal form equation, and show that the electromagnetic field can be described by a Ginzburg-Landau equation in a vector form. Then by using topological arguments we show the possibility of vectorial topological defects which are not predictable by the scalar theory.

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The semiclassical description of the laser dynamics brings into play the wave equation coming from the Maxwell theory for a dielectric and two other equations describing, respectively, the behavior of the dipole polarization and the inverted atomic population. Most of the papers dealing with lasers also assume that the direction of the electric field is constant, and reduce the dynamics to its scalar form. This assumption, widely justified by the common technological designs, together with the slowly varying envelope approximation, leads to the well-known Maxwell-Bloch set of equations [1]. Recently this model has been studied by normal form equations, new solutions have been predicted [2] and observed [3], and the relationship between nonlinear optics and the complex Ginzburg-Landau equation has been proved in a general way [4]. Now in the same way that the *optical vortices* predicted by Ref. [2] were a nonlinear version of the linear singularities discussed by Berry, we are interested here in the nonlinear version of the topological vectorial defects of an electromagnetic wave studied by Nye and Hajnal [5]. Therefore the aim of this paper is to derive a normal form equation for the Maxwell-Bloch model, keeping in mind the vectorial nature of both the electric field and dipole polarization, and to exhibit typical nonlinear vectorial solutions that cannot be predicted by the scalar theory. From a mathematical point of view, the computation we perform here is a true codimension-one bifurcation, bringing into play two complex order parameters which are covariant by space rotation.

The interaction of an electromagnetic field with the matter can be modeled by [6]

$$\begin{aligned}
\frac{\partial^2\mathbf{E}}{\partial t^2} &= -\mu_0 c^2\frac{\partial^2\mathbf{P}}{\partial t^2} + c^2[\nabla^2\mathbf{E}-\nabla(\nabla\cdot\mathbf{E})] - \mathcal{K}\frac{\partial\mathbf{E}}{\partial t}\,,\\
\frac{\partial^2\mathbf{P}}{\partial t^2} &= -\gamma_\perp\frac{\partial\mathbf{P}}{\partial t} - \omega_a^2\mathbf{P} - gD\mathbf{E}\,, \qquad (1)\\
\frac{\partial D}{\partial t} &= -\gamma_\parallel(D-D_0) + \frac{2}{\hbar\omega}\left[\mathbf{E}\cdot\frac{\partial\mathbf{P}}{\partial t}\right].
\end{aligned}$$

$\mathbf{E}$, $\mathbf{P}$, and D are real quantities which correspond, respectively, to the electric vector field, the dipole polarization vector, and the population inversion. Note that the equation for the polarization we have used is often referred to as the Lorentz model for an atom [7]. Although this equation is not derived from the quantum mechanics of two-level systems, but from a purely classical approach, it takes into account the most significant features of real atomic transitions [8]. Especially the fact that (1) can be reduced [6] via a slowly varying envelope approximation to the quantum mechanically correct Maxwell-Bloch set of equations ensures that both quantum and classical descriptions will lead, at least at leading order, to the same final results. In this model, μ_0 is the magnetic susceptibility, c is the speed of light, $\mathcal{K}$ is a damping term associated with the losses (including the mirror losses [9]), while $\gamma_\perp$ and $\gamma_\parallel$ are decay rates of the material variables. $\hbar\omega_a$ is the energy gap between the two atomic levels, g is a coupling constant between the electric field and the population inversion, and D_0 is the pumping term. *A priori* ω is the electric field frequency, very close to the atomic frequency, and in what follows we will assume $\omega=\omega_a$. Also we consider a laser with a transverse cross section large enough to allow the occurrence of spatial structures.

$\mathbf{E}=0$, $\mathbf{P}=0$, and $D=D_0$ is an obvious solution, stable for D_0 smaller than a critical value $D_{0c}=\mathcal{K}\gamma_\perp/\mu_0 c^2 g$. When $D_0=D_{0c}$, this basic solution is unstable with respect to traveling waves lasing with frequency ω_a and critical wave vector $k_c=\pm\omega_a/c$. The direction of propagation $\mathbf{k}_c/k_c$ is not defined, but we suppose that the geometry of the laser medium or mirrors selects the longitudinal direction (z) [10]. Considerations will be limited to only the right traveling waves ($\mathbf{k}_c=k_c\hat{\mathbf{z}}$), though *a priori* both directions of propagation are equiprobable (ring cavity model). As usual in nonlinear physics [10], a perturbative nonlinear analysis is performed near the laser threshold by introducing a small parameter defined by $D_0=D_{0c}+\epsilon^2\tilde{D}_0$ ($\epsilon\ll 1$). Then the development

$$\begin{pmatrix}\mathbf{E}\\ \partial_t\mathbf{E}\\ \mathbf{P}\\ \partial_t\mathbf{P}\\ D\end{pmatrix} = \begin{pmatrix}\mathbf{0}\\ \mathbf{0}\\ \mathbf{0}\\ \mathbf{0}\\ D_0\end{pmatrix} + \epsilon\begin{pmatrix}\mathbf{E}_1\\ \partial_t\mathbf{E}_1\\ \mathbf{P}_1\\ \partial_t\mathbf{P}_1\\ D_1\end{pmatrix} + \epsilon^2\begin{pmatrix}\mathbf{E}_2\\ \partial_t\mathbf{E}_2\\ \mathbf{P}_2\\ \partial_t\mathbf{P}_2\\ D_2\end{pmatrix} + \cdots,$$

with

$$\begin{pmatrix} \mathbf{E}_1 \\ \partial_t \mathbf{E}_1 \\ \mathbf{P}_1 \\ \partial_t \mathbf{P}_1 \\ D_1 \end{pmatrix} = \begin{pmatrix} \mathcal{A} \\ i\omega_a \mathcal{A} \\ \dfrac{i\mathcal{K}}{\mu_0 c^2 \omega_a}\mathcal{A} \\ -\dfrac{\mathcal{K}}{\mu_0 c^2}\mathcal{A} \\ 0 \end{pmatrix} e^{i(\omega_a t - k_c z)} + \text{c.c.}, \quad \mathcal{A} \perp \hat{\mathbf{z}},$$

and where $\mathcal{A}$ is slowly varying in space and time ($X=\epsilon x$, $Y=\epsilon y$, $Z=\epsilon^2 z$, $T=\epsilon^2 t$), is a solution of (1) up to order ϵ^3, provided that the following solvability condition is satisfied:

$$\partial_T \mathcal{A} = c_1 \mathcal{A} - c_2 \left[\partial_Z + \frac{i}{2k_c}\nabla_\perp^2\right]\mathcal{A} + c_3 \left[\partial_Z + \frac{i}{2k_c}\nabla_\perp^2\right]^2 \mathcal{A} - c_4(\mathcal{A}\cdot\mathcal{A}^*)\mathcal{A} - c_5(\mathcal{A}\cdot\mathcal{A})\mathcal{A}^* \tag{2}$$

with

$$c_1 = \frac{\mu_0 c^2 g \tilde{D}_0}{2(\mathcal{K}+\gamma_\perp)}, \quad c_2 = \frac{c\gamma_\perp}{\mathcal{K}+\gamma_\perp},$$

$$c_3 = \frac{\gamma_\perp c^2 [4\mathcal{K}\omega_a + i\gamma_\perp(\mathcal{K}+\gamma_\perp)]}{2\omega_a(\mathcal{K}+\gamma_\perp)^3},$$

$$c_4 = \frac{2\mathcal{K}g}{\hbar\omega_a\gamma_\parallel(\mathcal{K}+\gamma_\perp)},$$

$$c_5 = \frac{\mathcal{K}g(\gamma_\parallel - 2i\omega_a)}{\hbar\omega_a(\mathcal{K}+\gamma_\perp)(\gamma_\parallel^2+4\omega_a^2)}.$$

Equation (2) is the 3D vectorial Ginzburg-Landau equation ruling the behavior of the electric field in a dielectric. It looks like those obtained in hydrodynamics for the description of the slow space and time behavior of a convective pattern [10], except that it is now a vectorial equation and the nonlinear part is split into two terms [11]. This equation also possesses strong analogies with the vectorial nonlinear Schrödinger equation used in plasma physics to describe the envelope behavior of an electromagnetic wave [12].

Several further assumptions are needed to simplify Eq. (2). First, following the assumptions of Ref. [13] (i.e., uniform field limit and a large free spectral range), we can extract the longitudinal dependence of the amplitude A with

$$\mathcal{A}(X,Y,Z,T) = \mathcal{B}(X,Y,T)e^{-i(\Delta Z)}.$$

Second, for the sake of simplicity, we restrict ourselves to the case where Δ is positive such that the fourth-order derivative is not needed anymore (the case $\Delta<0$ gives qualitatively the same results but is more complex). With these hypotheses the normal form equation can be expressed as

$$\partial_t \mathcal{B} = z_1\mathcal{B} + z_2\nabla_\perp^2\mathcal{B} - z_3(\mathcal{B}\cdot\mathcal{B}^*)\mathcal{B} - z_4(\mathcal{B}\cdot\mathcal{B})\mathcal{B}^*, \tag{3}$$

where $z_1 = c_1 + i\Delta c_2 - \Delta^2 c_3$, $z_2 = (2c_3\Delta - ic_2)/2k_c$, $z_3 = c_4$, and $z_4 = c_5$.

Now we want to perform a general study of Eq. (3), without restricting ourselves to the realistic laser values of the coefficients z_i [14]. In what follows we only assume that the real parts of z_1, z_2, and z_3 are positive as this is already the case for a laser above the threshold. Then the usual scaling transformation [15] leads to

$$\partial_t \mathcal{B} = \mathcal{B} + (1+i\alpha)\nabla_\perp^2\mathcal{B} - (1+i\beta)(\mathcal{B}\cdot\mathcal{B}^*)\mathcal{B} - (\delta+i\eta)(\mathcal{B}\cdot\mathcal{B})\mathcal{B}^* \tag{4}$$

with $\mathcal{B}\perp\hat{\mathbf{z}}$. Now in order to be able to predict the existence of topological defects, a rough description of the geometry of the set of solutions ($\mathcal{S}$) is needed [16]. Symmetry considerations are helpful to provide a global characterization of $\mathcal{S}$. For example, in our case, because of the gauge and space rotation symmetries, we know that the set of solutions ($\mathcal{S}$) is globally invariant with respect to the following transformation:

$$\mathcal{B} = \begin{pmatrix} B_1 \\ B_2 \\ 0 \end{pmatrix} \in \mathcal{S} \Longrightarrow \begin{cases} G_\phi(\mathcal{B}) = e^{i\phi}\mathcal{B} \in \mathcal{S}, \\ R_\psi(\mathcal{B}) = \begin{pmatrix} \cos(\psi)B_1 + \sin(\psi)B_2 \\ -\sin(\psi)B_1 + \cos(\psi)B_2 \\ 0 \end{pmatrix} \in \mathcal{S}, \end{cases} \tag{5}$$

where ϕ and ψ are arbitrary real and constant values. Therefore, as

$$\mathbf{B}_\text{I}^\pm = \frac{1}{\sqrt{2}}e^{-i(\beta t)}\begin{pmatrix} 1 \\ \mp i \\ 0 \end{pmatrix}, \quad \mathbf{B}_\text{II} = \frac{e^{-i[(\beta+\eta)/(1+\delta)]t}}{(1+\delta)^{1/2}}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

are homogeneous time periodic solutions of (4), stable, respectively, for (I)

$$1+\alpha\beta>0, \quad \delta>0, \tag{6a}$$

and for (II)

$$\frac{\delta+\alpha\eta}{\delta}>0, \quad \frac{1+\alpha\beta+\delta+\alpha\eta}{1+\delta}>0, \quad \frac{2\delta}{1+\delta}<0, \tag{6b}$$

the sets $\mathcal{S}_\text{I}$ and $\mathcal{S}_\text{II}$ defined by

$$\mathcal{S}_i = \{G_\phi(R_\psi(\mathbf{B}_i)), (\phi,\psi) \in [0,2\pi]\times[0,2\pi]\}, \quad i=\text{I},\text{II},$$

are subsets of $\mathcal{S}$ with, respectively, the circle $[G_\phi(\mathcal{S}_\text{I}) = R_\phi(\mathcal{S}_\text{I})]$ and the torus topology. Note that they correspond, respectively, to a circular and a straight polarized electric wave,

$$G_\phi(\mathbf{B}_\text{I}^\pm) \rightarrow \mathbf{E}_\text{I}^\pm = \frac{1}{\sqrt{2}}\begin{pmatrix} \cos[(\omega_a-\beta)t - (k_c+\Delta)z + \phi] \\ \pm\sin[(\omega_a-\beta)t - (k_c+\Delta)z + \phi] \\ 0 \end{pmatrix},$$

$$G_\phi(R_\psi(\mathbf{B}_{\mathrm{II}})) \to \mathbf{E}_{\mathrm{II}} = \frac{\cos\{[\omega_a - (\beta+\eta)/(1+\delta)]t - (k_c+\Delta)z + \phi\}}{(1+\delta)^{1/2}} \begin{pmatrix} \cos(\psi) \\ -\sin(\psi) \\ 0 \end{pmatrix} .$$

Topological defects are associated with phase transitions and are characterized by their asymptotic connection with solutions which can be exchanged by symmetry transformations [16]. For example, a point topological defect with a core localized at point P, and associated with $\mathcal{S}_{\mathrm{II}}$ and the phase ϕ, is a solution of Eq. (4) which connects asymptotically some elements of $\mathcal{S}_{\mathrm{II}}$ in such a way that the phase ϕ used to generate $\mathcal{S}_{\mathrm{II}}$ varies by $+2\pi$ (-2π) along any closed line (Γ) surrounding the point P (Fig. 1(c) [Fig. 2(c)]). In what follows we will use the notation $(\mathcal{S}_{\mathrm{II}},\phi,+)$ $[(\mathcal{S}_{\mathrm{II}},\phi,-)]$ to refer to such a defect. Far from this defect core, a good analytical approximation of the electric field is then given by

$$\mathbf{E} \simeq \frac{\cos\{[\omega_a - (\beta+\eta)/(1+\delta)]t - (k_c+\Delta)z + \Phi(r,\theta,t)\}}{(1+\delta)^{1/2}} \times \begin{pmatrix} \cos(\psi_0) \\ -\sin(\psi_0) \\ 0 \end{pmatrix}, \text{ with } \oint_\Gamma \nabla\Phi\cdot d\mathbf{r} = 2\pi ,$$

where (r,θ) are the polar coordinates associated with the point P, and ψ_0 an arbitrary constant [we have used $\psi_0 = 4\pi/11$ in Figs. 1(c) and 2(c)]. When the phase ψ is involved in the definition of the defect, as for example in $(\mathcal{S}_{\mathrm{II}},\psi,+)$ [Fig. 1(d)], the corresponding analytical expression is then

$$\mathbf{E} \simeq \frac{\cos\{[\omega_a - (\beta+\eta)/(1+\delta)]t - (k_c+\Delta)z + \phi_0\}}{(1+\delta)^{1/2}} \times \begin{pmatrix} \cos[\Psi(r,\theta,t)] \\ -\sin[\Psi(r,\theta,t)] \\ 0 \end{pmatrix}, \text{ with } \oint_\Gamma \nabla\Psi\cdot d\mathbf{r} = 2\pi .$$

The exact shapes of the functions Ψ and Φ cannot be obtained from topological arguments because they depend on the defect core [17]. It is worth noting that, as $\mathcal{B}$ is a very smooth function (obeying a partial differential equation), each defect core, which is either a phase ϕ or ψ singularity, must correspond to a vanishing of the modulus of the electric field. There is then no singularity for $\mathbf{E}$ and no magnetic monopole.

A priori, using the sets of solutions $\mathcal{S}_{\mathrm{I}}^{\pm},\mathcal{S}_{\mathrm{II}}$ and the symmetry transformations (gauge, rotation, gauge+rotation) of Eq. (4), twelve distinct topological defects can be built. They are

$$(\mathcal{S}_{\mathrm{I}}^{\pm},\phi,\pm),\quad (\mathcal{S}_{\mathrm{II}},\psi,\pm),\quad (\mathcal{S}_{\mathrm{II}},\phi,\pm),\quad (\mathcal{S}_{\mathrm{II}},\phi,\pm,\psi,\pm).$$

It is usual to group the defects by pairs (defect-antidefect), corresponding, respectively, to a phase circulation equal to 2π and -2π. From a theoretical point of

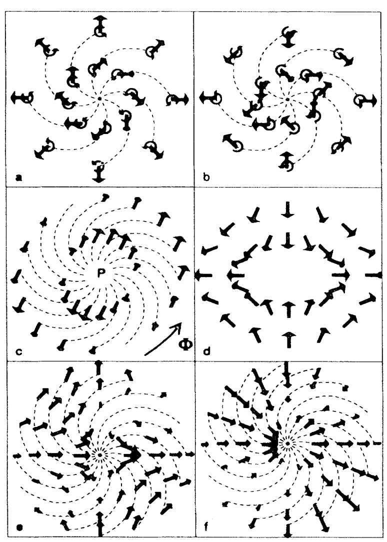

FIG. 1. Defect solutions: (a) $(\mathcal{S}_{\mathrm{I}}^{+},\phi,+)$, (b) $(\mathcal{S}_{\mathrm{I}}^{-},\phi,+)$, (c) $(\mathcal{S}_{\mathrm{II}},\phi,+)$, (d) $(\mathcal{S}_{\mathrm{II}},\psi,+)$, (e) $(\mathcal{S}_{\mathrm{II}},\psi,+,\phi,+)$, and (f) $(\mathcal{S}_{\mathrm{II}},\psi,-,\phi,+)$. For type-I defects, we have chosen t and z such that $(\omega_a-\beta)t-(k_c+\Delta)z=0$, while for type-II defects $[\omega_a-(\beta+\eta)/(1+\delta)]t-(k_c+\Delta)z=0$. In all the pictures, the straight black arrows stand for the electric field vector, and the dashed curves with a spiral shape are isophase ϕ lines. The length of the arrow is proportional to the norm of the electric field, but consideration has been limited to the region far from the defect core such that the vanishing of $|\mathbf{E}|$ at the core is not visible. For (a) and (b) the time evolution has been represented by the small arrows with a circular arc shape. For (c), (d), (e), and (f), the direction of the electric field is stationary; only the projection oscillates with time (see text).

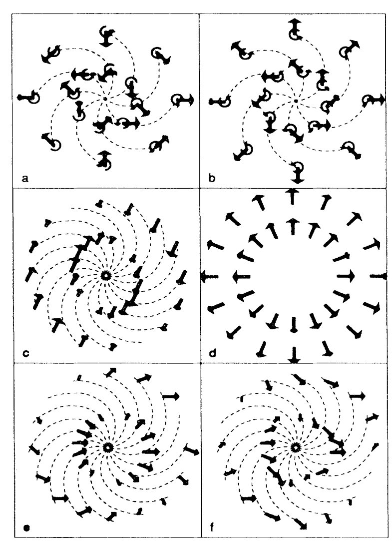

FIG. 2. Antidefect solutions: (a) $(\mathcal{S}_{\mathrm{I}}^{+},\phi,-)$, (b) $(\mathcal{S}_{\mathrm{I}}^{-},\phi,-)$, (c) $(\mathcal{S}_{\mathrm{II}},\phi,-)$, (d) $(\mathcal{S}_{\mathrm{II}},\psi,-)$, (e) $(\mathcal{S}_{\mathrm{II}},\psi,-,\phi,-)$, and (f) $(\mathcal{S}_{\mathrm{II}},\psi,+,\phi,-)$.

view, this regrouping is associated with the creation and annihilation process, but in our case, because Eq. (4) is invariant under a parity symmetry transformation, this classification gives rise to some practical applications. Indeed with the parity transformation $(x,y) \rightarrow (x,-y)$, a phase circulation of $\pm 2\pi$ is exchanged into $\mp 2\pi$, that is to say, that a defect solution is exchanged into an antidefect one. This is the simple geometrical transformation which has been used to deduce the antidefect solutions of Fig. 2 from the defect pictures of Fig. 1. Note that, as might be expected, we recover [Fig. 1(c) or Fig. 2(c)] the topological solution called an *optical vortex* which was previously predicted in the framework of the scalar theory. Using a second-order space and time finite-difference scheme [18] and periodic boundary conditions, we have simulated Eq. (4) on a massive parallel computer (Connection Machine), and have numerically checked that all these topological solutions do exist for the range of parameters for which the asymptotically connected solutions are stable [i.e., conditions (6)]. Now three remarks are in order. First the higher-order terms which have been neglected in (2) or (4) may be important. However, as they are invariant with respect to the symmetry transformations (5), they do not modify the topology of $\mathcal{S}$ and therefore the existence of topological defects. Only the nature of the asymptotic solutions may be affected. Second, in the same way that it has recently been shown that the classical optical vortex can be understood in terms of Gauss-Laguerre modes [19], it may be possible to analyze some of the vectorial defects in terms of the usual waveguide modes [for example, Fig. 1(e) has similarities with the TM_{01} mode]. Third, though all these various defects correspond to dark spots in a time-averaged intensity plot, it should be possible to identify and characterize them experimentally, by interference of the defect field with either a linearly or a circularly polarized plane wave.

In conclusion, we have been able to reduce Eqs. (1) to their normal form, vectorial Ginzburg-Landau equation near the threshold of the lasing action and we have proved the importance of this description by exhibiting some beautiful and interesting solutions which cannot be predicted by the scalar theory. However, there are still many open questions. For example, analogous to the usual phase ϕ Benjamin-Feir instability regime [20], we can expect the vectorial equation to possess a phase ψ instability, characterized by the lack of correlation in the direction of the electric field. Another question is the interaction between distinct defects and their respective role in the two-phase instability regimes. Work in this direction is in progress.

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